Merrill Lynch London Equity Derivatives Seminar

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Option Theory

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&
Network Models

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OVERVIEW

1. Underlying price dynamics
- Stocks / Indices / Commodities.

2. Modelling dynamics by state graphs
- Trees / Lattices

3. Pricing an option over one period
- Replicating & hedging portfolios / Risk-neutral probs

4. Arbitrage
- No-arbitrage theorem / Pricing an option by arbitrage

5. Complete and incomplete markets
- Pricing in incomplete markets

6. Multiple periods
- State recombination / Lattices / Pricing options on lattices

7. Black and Scholes
- Put-Call parity / Black&Scholes formula / The Greeks

8. Summary, Conclusions, State-of-the-Art

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What is an Option?

An Option is a contract on an **underlying**.

Therefore, understanding and modelling the behaviour of the underlying is of paramount importance in pricing and hedging an option.

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PART 1

UNDERLYING PRICE DYNAMICS

An underlying that exhibits growth

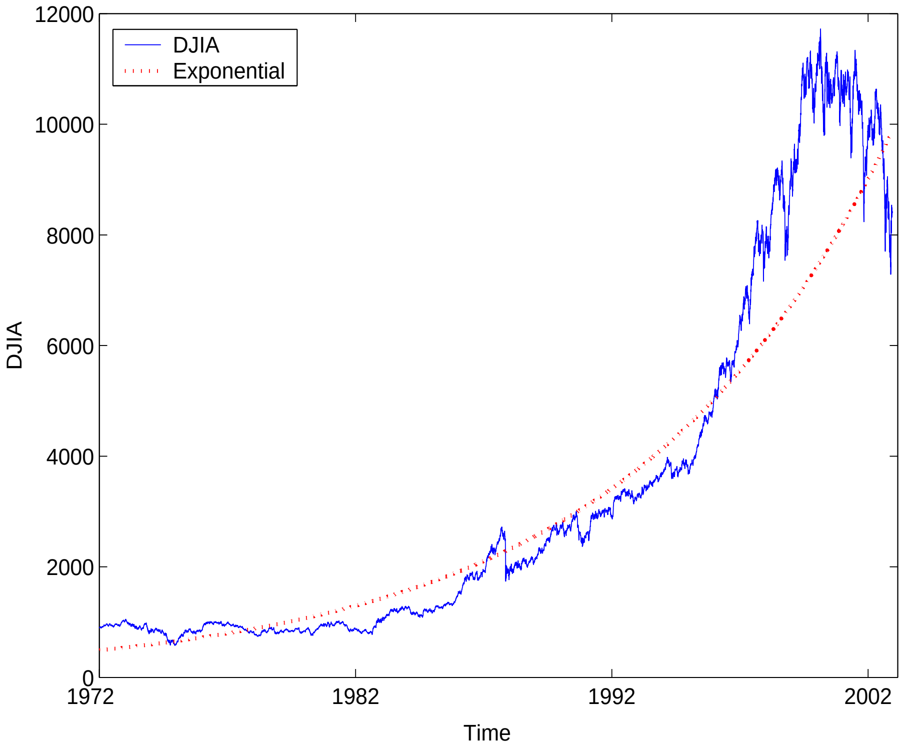

Figure 1: Growth of DJIA over 30 years

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Distribution of returns

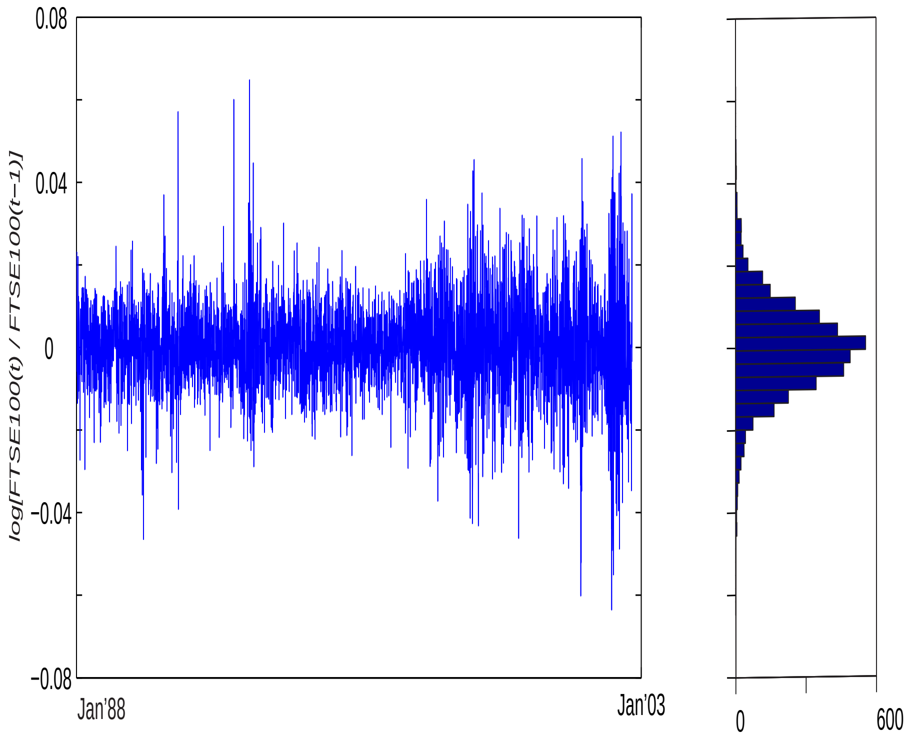

Figure 2: Daily returns and their distribution

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Random Walk

Even in a small time interval $[t, t + dt]$ the DJIA index exhibits both *growth* and *randomness*.

Growth is modelled by *continuous compounding*.

Randomness, on the other hand, is introduced by a special 'variable' called a *Random Walk*.

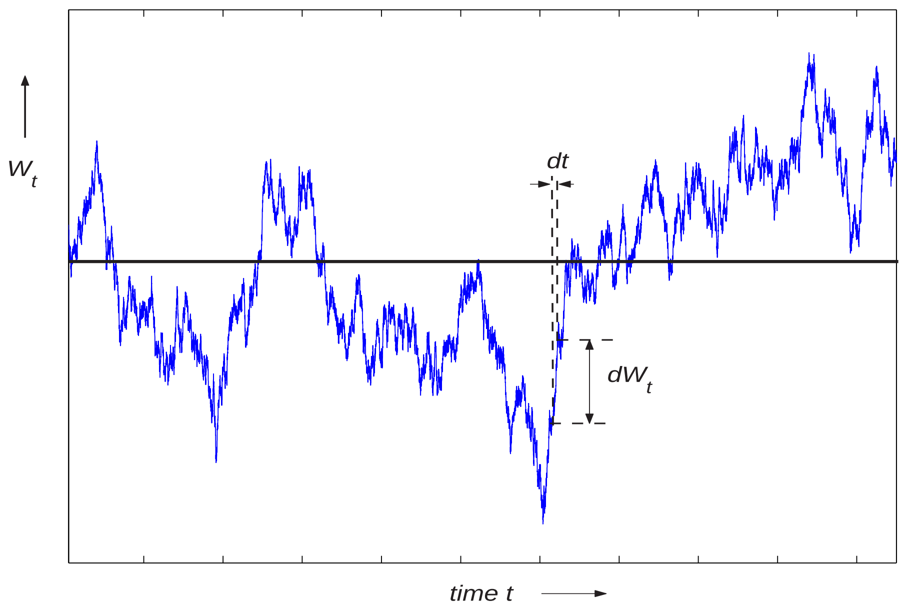

Figure 3: One-dimensional random walk.

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Random Walk

The symbol dW_t is used for the change in this variable during $[t, t + dt]$.

dW_t is supposed to account for the uncertainty in the DJIA changes during $[t, t + dt]$.

Essentially $dW_t = \sqrt{dt} \cdot \epsilon_t$ where ϵ_t is a random sample drawn from a Normal distribution with mean 0 and variance 1.

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Geometric Brownian Motion (GBM)

GBM is a reasonable representation of the dynamics of an equity's price x_t.

The equity return over the time interval $[t, t + dt]$ is given by

$$\frac{dx_t}{x_t} = \mu dt + \sigma dW_t$$

μ is the **drift**, σ is the **volatility**,
both annualised and assumed constant.

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Geometric Brownian Motion (GBM)

The price of the asset at time t is

$$x_t = x_0 \cdot e^{[\tilde{\mu}t + \sigma\sqrt{t}\,\epsilon]}$$

where x_0 is the initial price at time $t = 0$ and $\tilde{\mu} = \mu - \sigma^2/2$.

The pdf of the equity return at time t is a Normal distribution.
The pdf of x_t at time t is a Lognormal distribution.

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Fat-tails of return distribution

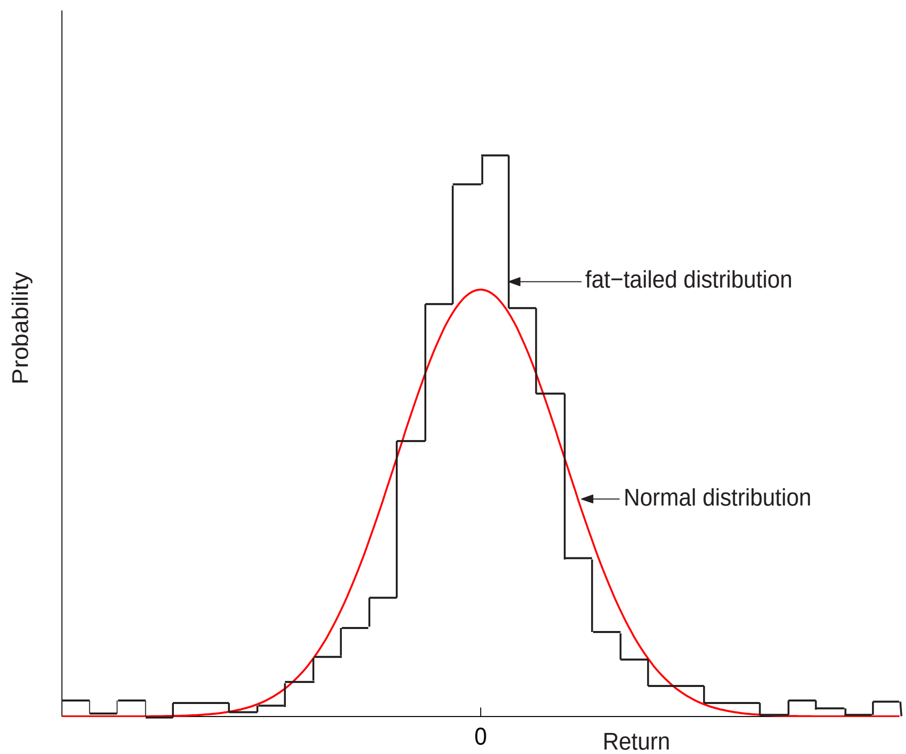

Figure 4: Asset returns are leptokurtic.

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Geometric Brownian Motion (GBM)

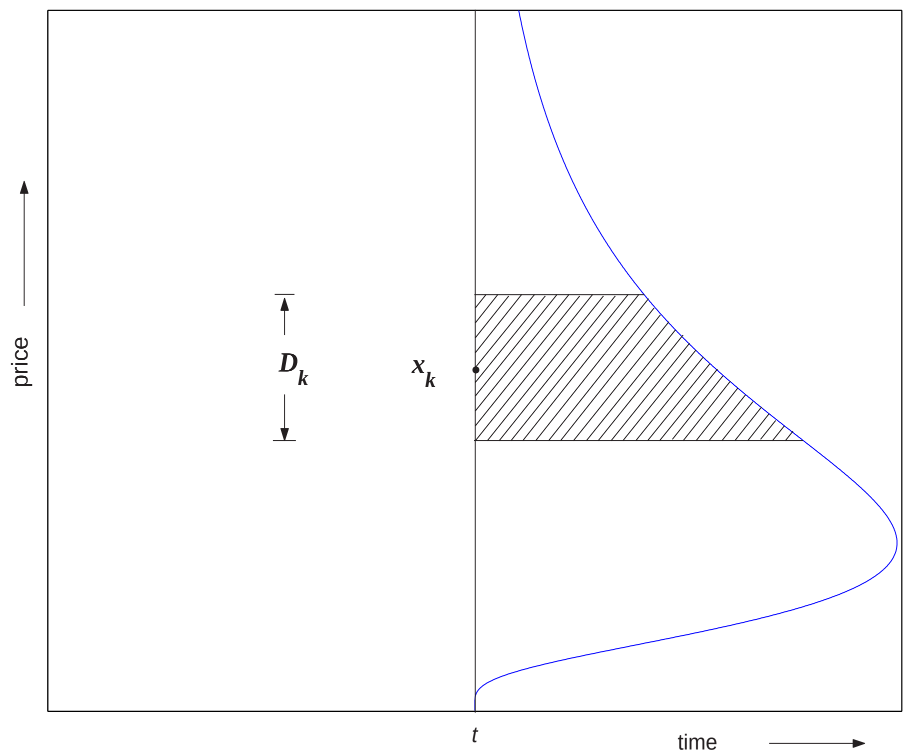

Figure 5: A lognormal price distribution.

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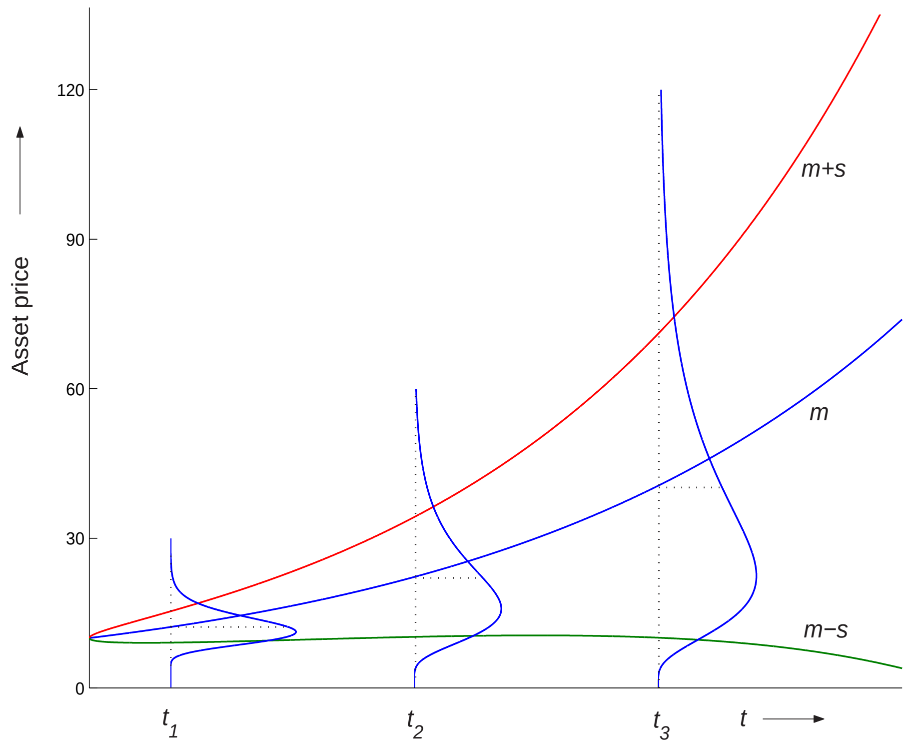

Figure 6: Evolution of the probability density function with time

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An underlying exhibiting Mean Reversion

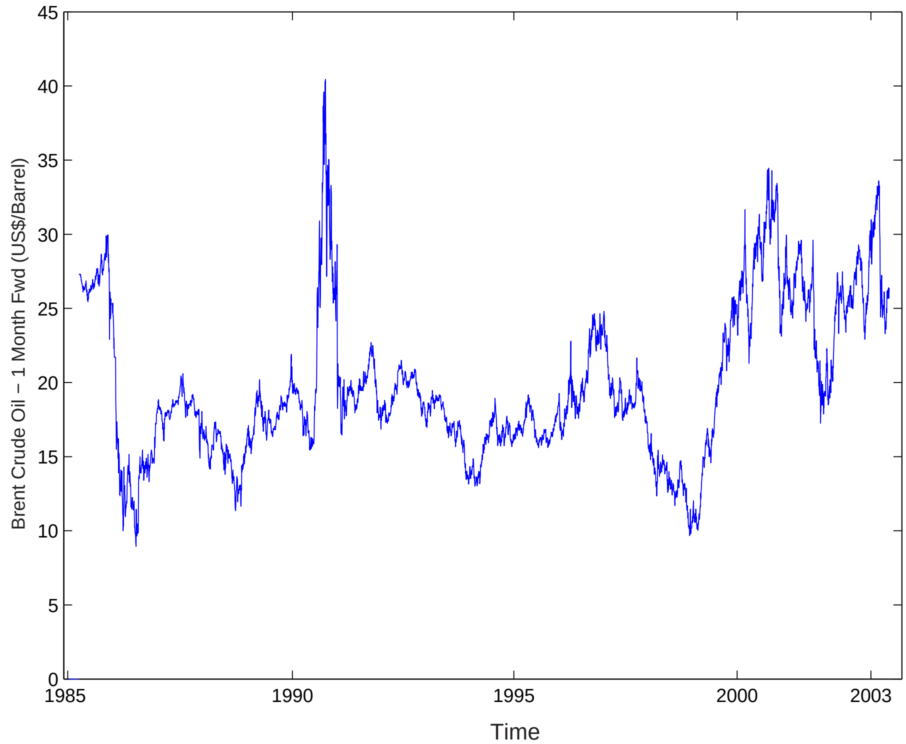

Figure 7: Brent crude price over 18 years.

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Mean Reverting Process (MR)

MR is a reasonable representation of the dynamics of the oil price x_t.

The price change over the time interval $[t, t + dt]$ is given by

$$dx_t = \mu(\bar{x} - x_t)dt + \sigma dW_t$$

$\bar{x}$ is the average price of oil over time, i.e. the **reversion level** to which the oil price tends.
μ is the **reversion rate**, i.e. the strength of the pull on the price towards the reversion level.
σ is the **volatility** as before.

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Tendency to the average in a Mean Reverting process

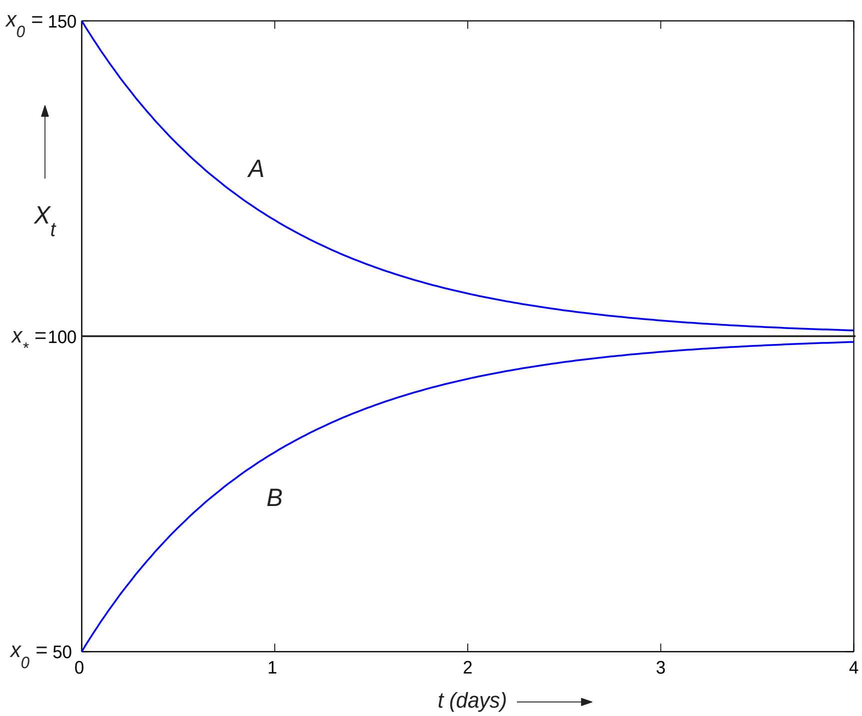

Figure 8: Mean Reversion

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PART 2

MODELLING DYNAMICS BY STATE GRAPHS

(Single Period)

Approximation of a price pdf

Rather than deal with a continuous price distribution, it is easier to approximate the pdf by a set of discrete prices and corresponding occurrence probabilities of these prices.

To be a reasonable approximation, we want the discrete prices to have the same mean and variance as the continuous distribution.

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Binary approximation

Two discrete prices a_1 and a_0 and their corresponding probabilities as q_1 and q_0.

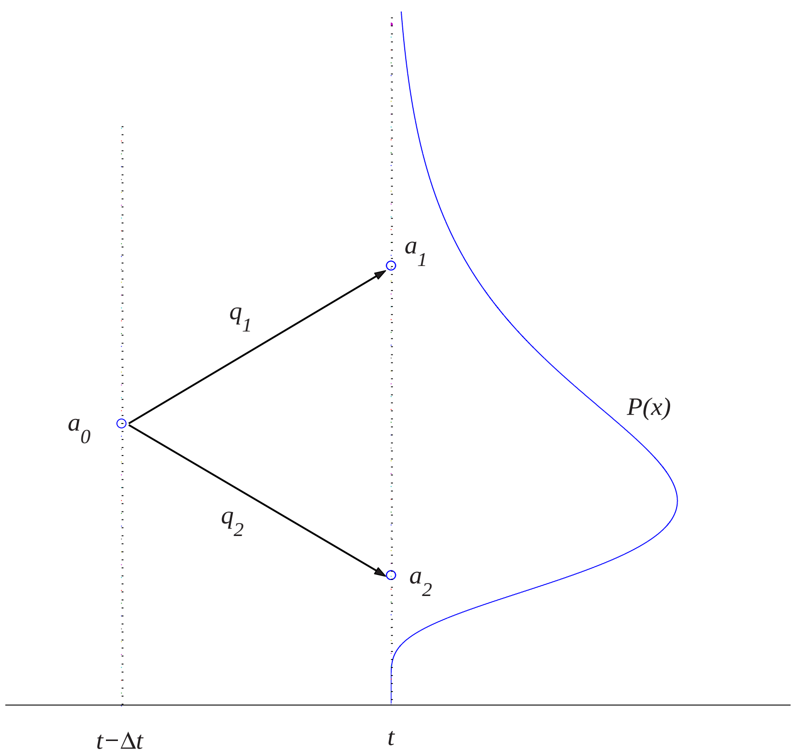

Figure 9: Approximation of a pdf with two values.

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Approximation of a price pdf

Assume we know the mean m and standard deviation s of the pdf.
In general there is more than one possible approximation

Approximation with equal probabilities. Setting $q_1 = q_2 = 1/2$ produces

$$a_1 = m + s, \quad a_2 = m - s$$

If $s > m$ then $a_2 < 0$ which may be infeasible.

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Approximation of a price pdf

Approximation with geometric symmetry about m.

Consider a factor $u > 1$ and assume $a_1 = um$, $a_2 = m/u$. The solution is:

$$q_1 = \frac{1}{u+1}, \qquad q_2 = \frac{u}{u+1}$$

$$u = \alpha + \sqrt{\alpha^2 - 1}$$

where $\alpha = 1 + \frac{1}{2}\left(\frac{s}{m}\right)^2$.

We can then compute a_1, a_2, q_1, q_2, feasible in *all* circumstances.

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PART 3

PRICING AN OPTION

(Single Period)

Pricing a call option

Assume two assets: asset 1 is a stock, asset 2 is a money market (MM) account.
The figure below shows the prices of one *unit* of the two assets. State 0 is now. States 1 and 2 are the only two possible states a year from now. The price of asset 1 in one year is lognormally distributed but we approximated its pdf with just two equally likely values (€120 and €95). The MM account pays a certain 5% pa.

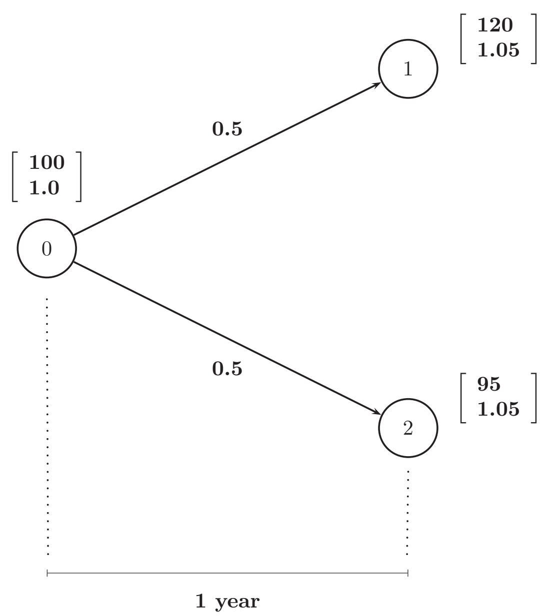

Figure 10: One period binary tree approximation to asset dynamics.

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Pricing a call option

For a 1-year call option with strike €108, the option payoff at states 1 and 2 are €12 and €0 respectively.
What is a fair value for this option?

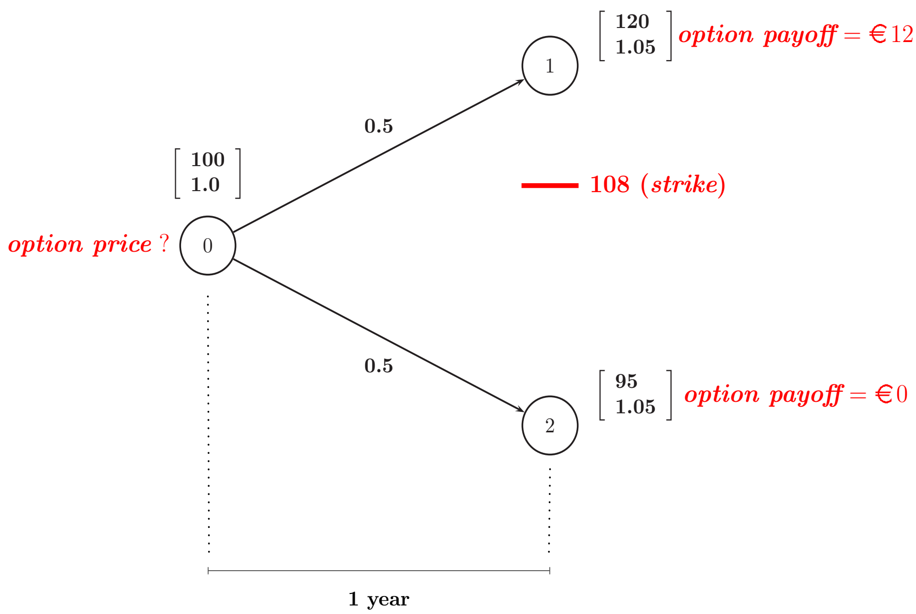

Figure 11: Option payoffs

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Pricing a call option

Naive approach that is WRONG

The expected payoff of the option in 1 year is

$$\frac{1}{2} \times €12 + \frac{1}{2} \times €0 = €6$$

Discounted to now becomes 6/1.05 = €5.71.

What is wrong with this argument?

The argument would give the same answer for a **certain** payoff of €6 in 1 year as for the option above. In other words no allowance has been made of the fact that the option payoff involves **risk**.

The market price of risk has been ignored.

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Pricing a call option

Correct approach is based on **arbitrage** considerations, specifically on the *Law of one price* which states:

If two assets (or portfolios) have exactly the same payoffs in the future, they must be worth the same now.

Using our two assets let us try to create a portfolio whose payoffs exactly replicate the option's payoffs.

Buy $y(1)$ units of asset 1 and $y(2)$ units of asset 2. We have:

$$\begin{aligned} &\text{for state 1:} \quad 120y(1) + 1.05y(2) = 12 \\ &\text{for state 2:} \quad 95y(1) + 1.05y(2) = 0 \end{aligned}$$

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Pricing a call option

The solution is: [$y(1) = 0.48$ and $y(2) = -43.429$].

For an investor, it is immaterial whether he holds this portfolio or the option.

The price of this portfolio now is:

$$€\,100 \times 0.48 + €\,1.0 \times (-43.429) = €\,4.57$$

and, therefore, this is also the option price.

The option seller can:

- Sell the option for €4.57
- Sell short 43.429 of asset 2 (namely, borrow €43.43 at 5%)
- Buy 0.48 units of asset 1 (and pay €48.00)

His position is completely without risk.

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Observations on pricing a call option

- The probabilities of occurrence of states 1 and 2 did not enter into the option price computation.
- The expected rate of growth of the stock (which is 7.5 % in the example) did not enter into the option price computation.
- The risk free interest rate (5 % on the MM account) is important in option pricing.

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Risk neutral probabilities

Consider the same example but with the prices indicated by symbols rather than numbers.

a is the stock price in state 0, ha and ℓa are the stock prices in states 1 and 2 respectively, where h and ℓ are two given ‘growth’ factors.
The MM unit is still worth 1 in state 0 and R in states 1 and 2.
Option payoffs: b_1 in state 1 and b_2 in state 2.

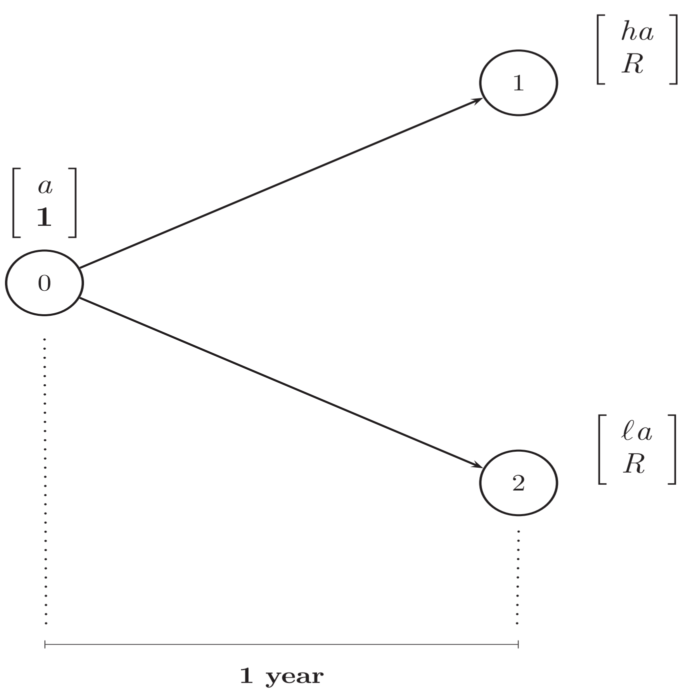

Figure 12: Computation of risk-neutral probabilities.

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Risk neutral probabilities

The replicating portfolio $\begin{bmatrix} y(1) \\ y(2) \end{bmatrix}$ must satisfy for states 1 and 2, respectively:

$$\begin{aligned} hay(1) + Ry(2) &= b_1 \\ \ell ay(1) + Ry(2) &= b_2 \end{aligned}$$

The solution is $y(1) = \frac{b_1 - b_2}{a(h-\ell)}, \quad y(2) = \frac{1}{R}\frac{hb_2 - \ell b_1}{h-\ell}$
and the option price:

$$P = \frac{b_1}{R}\left[\frac{R-\ell}{h-\ell}\right] + \frac{b_2}{R}\left[\frac{h-R}{h-\ell}\right]$$

Let us assume $h > R > \ell$ (we explain this later).

The expressions in [] are like probabilities, they are positive and add up to 1.

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Risk neutral probabilities

Let $\pi_1 = \left[\frac{R-\ell}{h-\ell}\right]$ and $\pi_2 = \left[\frac{h-R}{h-\ell}\right]$.
Then

$$P = \frac{1}{R}(b_1\pi_1 + b_2\pi_2)$$

i.e. The price of the option is equal to the "expected future payoff" discounted to the present by the risk free rate.

The expectation above is computed using **not the real** occurrence probabilities of the future states (both $\frac{1}{2}$ in the example), but the artificial occurrence probabilities π_1 and π_2.
In the earlier numerical example: $\pi_1 = 0.4$ and $\pi_2 = 0.6$ and $P = \frac{1}{1.05}(€12 \times 0.4 + €0 \times 0.6) =$ €4.57

π_1 and π_2 are called the **risk-neutral** probabilities. Note that they are a function of the underlying assets and are *independent* of the option.

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PART 4

ARBITRAGE

Arbitrage

Arbitrage is certainly the **most important concept** in finance.

It is deceptively simple but has widespread implications not only for option pricing/hedging but in every other quantitative finance area.

- A portfolio worth 0 at every possible future state, is worth 0 now.
- A portfolio with no liabilities at any possible future state and a possible positive worth at some state, has positive worth now.

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Arbitrage

The importance of these ‘obvious’ statements derives from the fact that if they are not satisfied, it would be possible for an investor to start with no money at all and *with certainty* accumulate infinite wealth, clearly an absurdity!

Thus, first and foremost, any proposed dynamics of financial asset price movements must satisfy the no-arbitrage conditions.

We will see (by example) that seemingly very reasonable assumptions are in fact not at all reasonable.

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Arbitrage

Consider the general 1-period evolution of $\hat{n}$ assets.

$a_j(n)$ is the price of asset n ($n = 1. \ldots, \hat{n}$) at state j ($j = 0, \ldots, \hat{j}$).

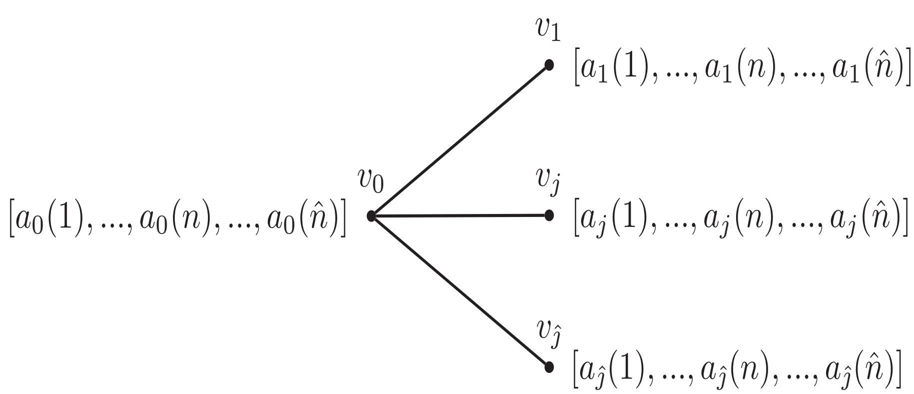

Figure 13: A general price evolution over one period.

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Arbitrage

No arbitrage theorem

The evolution of asset prices admits no arbitrage **if and only if** there exists a set of *positive* 'dual multipliers' u_j (one for each future state $j = 1, \ldots, \hat{j}$) such that for every asset n we have:

$$a_0(n) = u_1 a_1(n) + u_2 a_2(n) + \ldots + u_j a_j(n) + \ldots + u_{\hat{j}} a_{\hat{j}}(n)$$

The multipliers u_j can be normalised into risk-neutral probabilities:

$$\pi_j = \frac{u_j}{\sum_j u_j}$$

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Arbitrage

Revisiting the earlier example

We can easily verify that $u_1 = 0.38095$ and $u_2 = 0.57143$ satisfy the no-arbitrage conditions:
€120× 0.38095 + €95× 0.57143 = €100
€1.05× 0.38095 + €1.05× 0.57143 = €1.0

However, with a risk free rate of, say, 22% (i.e. $R = 1.22$ outside the range [0.95, 1.20]) then no two such positive values u can be found, because the suggested dynamics are then unreasonable (arbitrage exists).

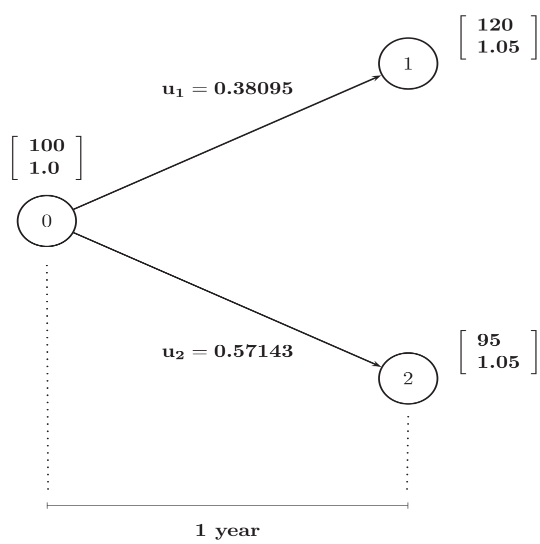

Figure 14: Proof of no arbitrage in example.

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Arbitrage

Arbitrage is easy to verify for this simple example, but this is not generally the case.

Producing a *state transition graph* with several correlated assets; with good approximation to the multidimensional pdf (i.e. with the correct means, variances and correlations) and at the same time ensuring that no arbitrage exists, is a very hard problem.

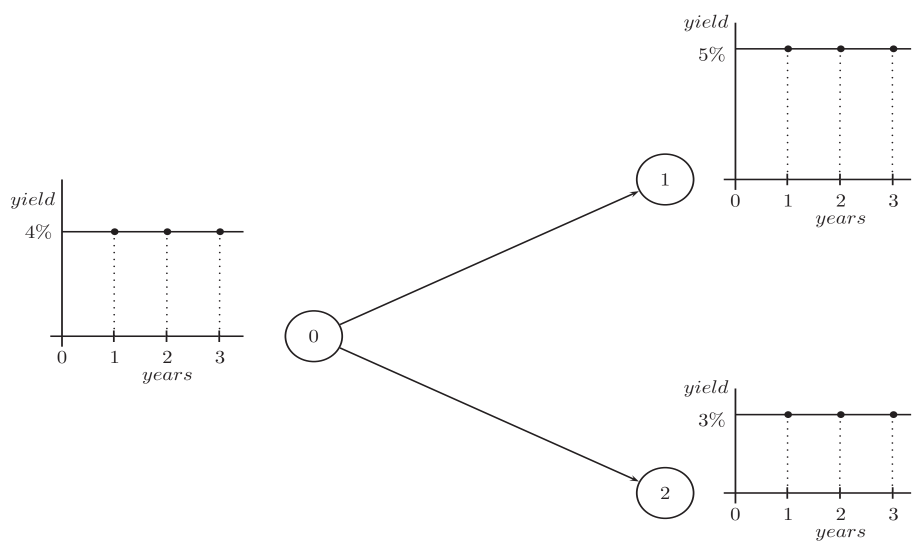

Figure 15: One period binary tree containing arbitrage.

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PART 5

COMPLETE & INCOMPLETE MARKETS

Complete Market

A market in which any payoff structure can be replicated by a portfolio of existing assets is called a **Complete Market**.

In a complete market all risk can be eliminated by hedging.

In a complete market "no-arbitrage" is equivalent to "the multipliers u are unique".

This then implies that an asset with a given payoff structure has a **unique price**. Any deviation from this price leads to arbitrage opportunities.

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Pricing options in a complete market

We have considered a set of **base** assets (2 in our example) forming a complete market. We derived the (unique) set of *dual multipliers* u corresponding to the no-arbitrage conditions. Any new asset that is introduced into the set of base assets must be *compatible* with u in order to avoid introducing arbitrage.

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Pricing options in a complete market

Given the future payoffs of the new asset (if it is an option, for example) the u's can be applied to find its current option price.

For the example, an option with payoffs of €12 and €0 in states 1 and 2 respectively, will be priced at:
€12$\times$ 0.38095 + €0$\times$ 0.57143 = €4.57
agreeing with the earlier price. The advantage now is that the pricing mechanism is general, does not require a risk-free rate to be known for example. It simply requires a derivative, say, not to introduce arbitrage when added as an 'asset' to the base class containing the option's underlying.

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Incomplete Market

A market in which some payoff structures cannot be replicated by a portfolio of existing assets is called an **Incomplete Market**.

In an incomplete market the risk **cannot** be totally eliminated by hedging.

In an incomplete market many sets of multipliers u satisfy the no-arbitrage conditions. The price of an asset with a given payoff structure is not uniquely determined.

The price can lie within a range without arbitrage opportunities arising.

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Incomplete Market

Market incompleteness arises because, for example:

- The underlying is not tradable.
  (weather derivatives, options on macro-economic variables)
- Lack of liquidity in the market.

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Pricing an option in an Incomplete Market

The price of an option with a given payoff structure is determined within the no-Arbitrage band by an equilibrium argument.

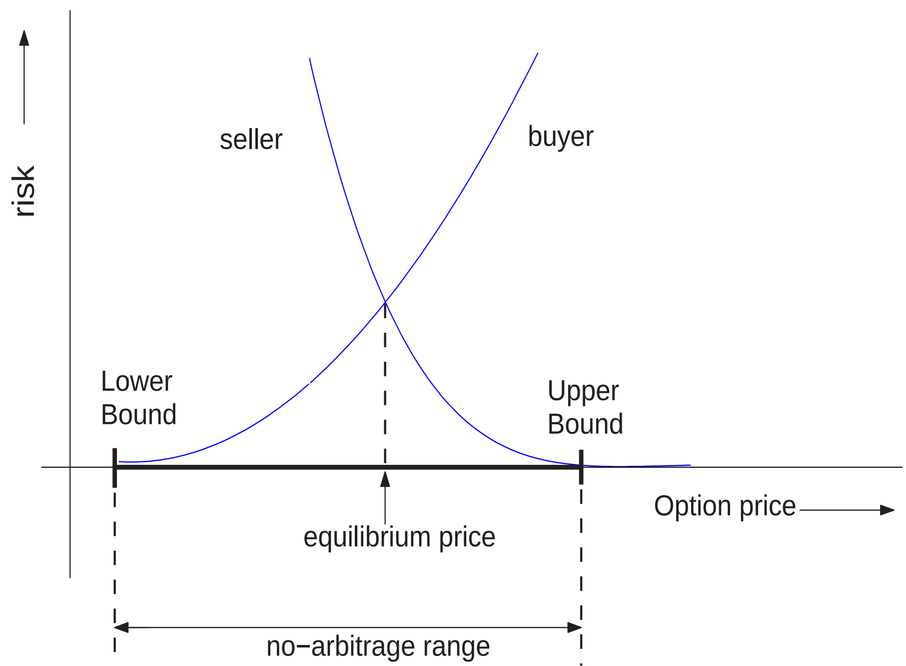

Figure 16: Equilibrium pricing in an incomplete market.

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PART 6

MULTIPLE PERIODS

Option pricing using multiple periods

By dividing the horizon (time to expiration) into more and more periods, a better approximation to the actual pdf of the underlying results and more accurate option prices are obtained.

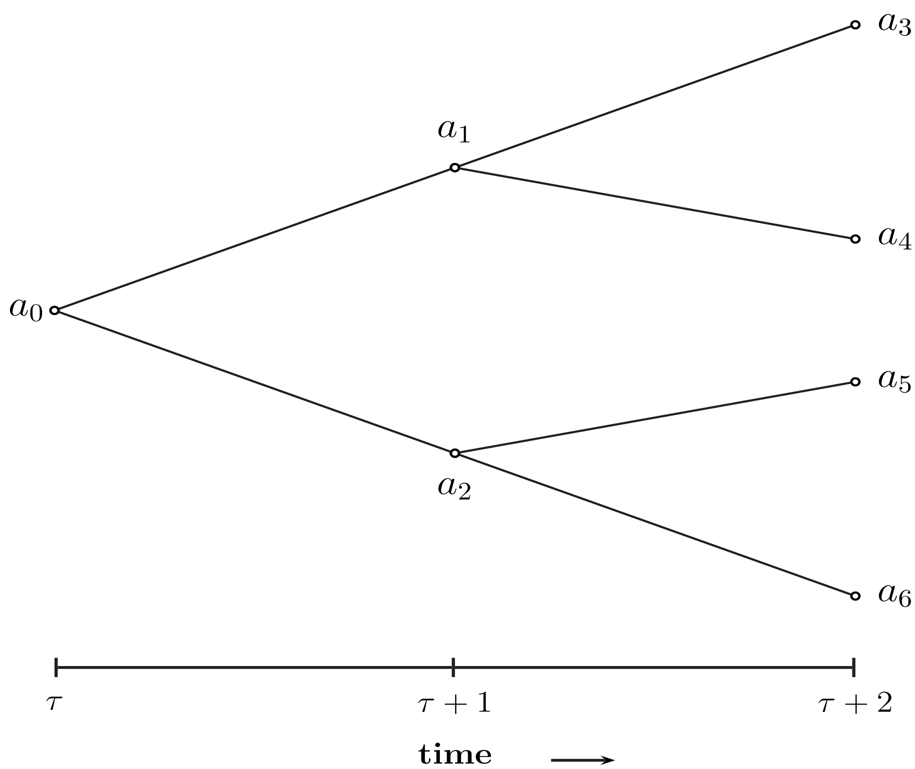

Figure 17: A 2-period tree.

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State recombination and lattices

The number of vertices generated at time τ is reduced if (accidentally or otherwise), state recombination takes place. Lattices are the result.

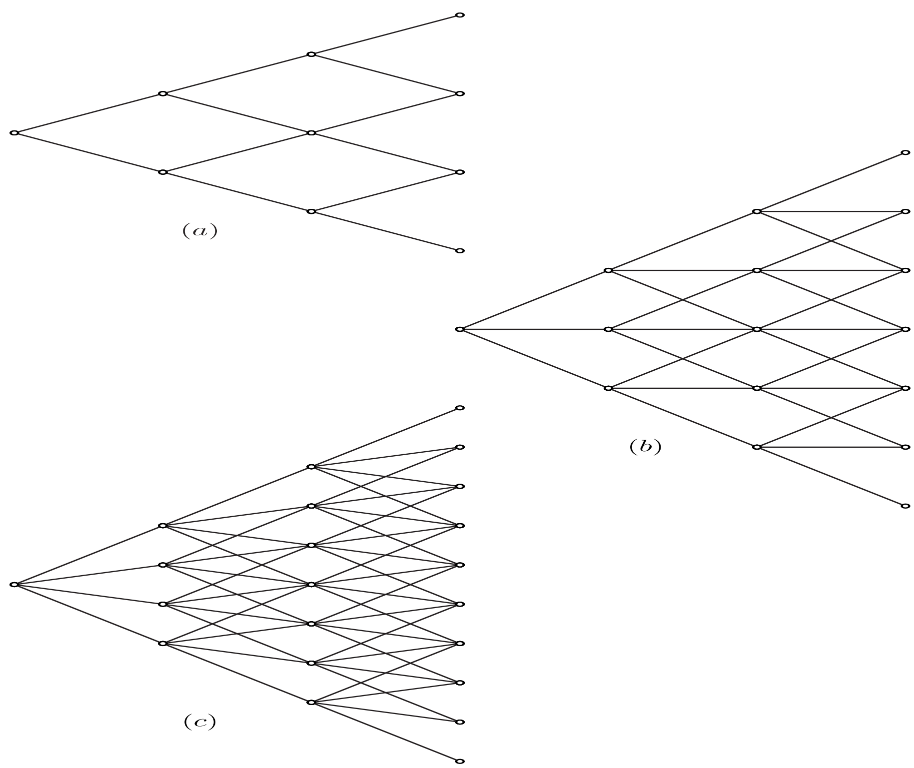

Figure 18: Lattices resulting from state recombination

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Previous example over 4 periods.

The stock in the 1-period example exhibited (over 1 year):

mean= €107.5, **standard deviation**= €12.5

Assuming the stock follows GBM, we can compute parameters

$\mu = 0.072321, \quad \sigma = 0.115889$

Over a quarter year period, this GBM produces a lognormal distribution which can be approximated by two states both with probabilities $\frac{1}{2}$ and values

$h = 1.077296$, and $\ell = 0.959194$

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Previous example over 4 periods.

The resulting lattice is:

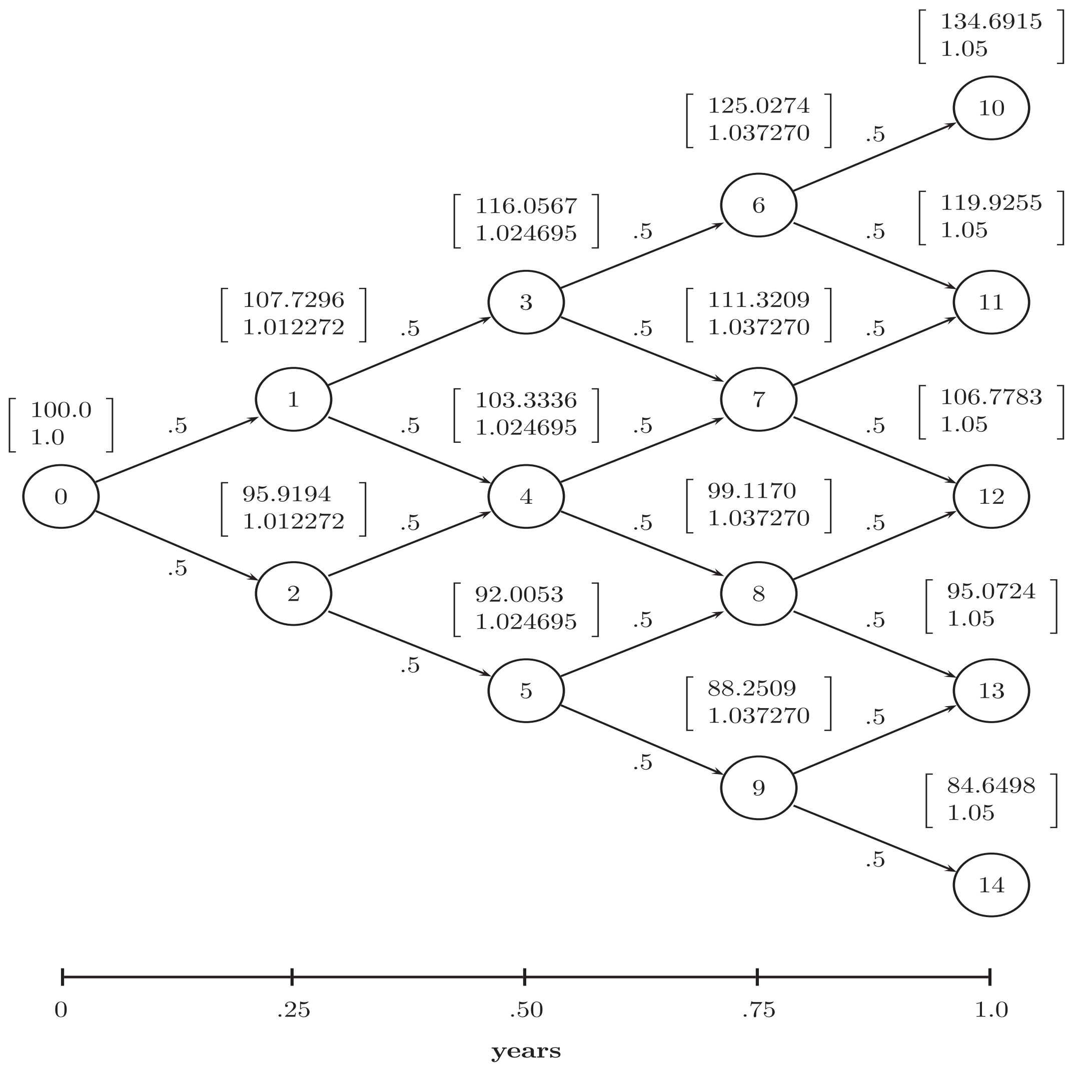

Figure 19: Example over 4 periods.

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Previous example over 4 periods.

The dual multipliers are:
$u_1 = 0.443996$ and $u_2 = 0.543881$
(for the 'up'/'down' moves from any state).

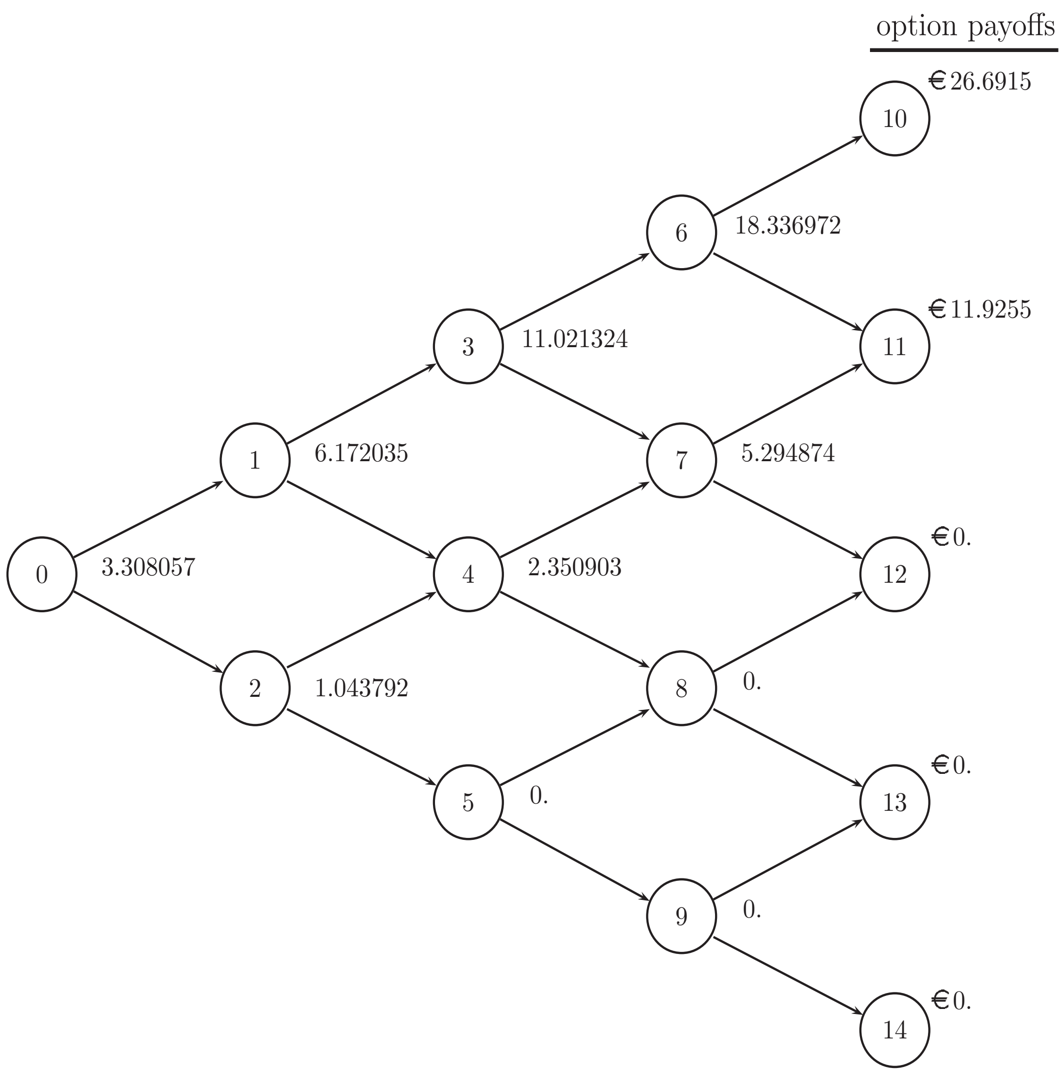

Figure 20: Example over 4 periods.

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PART 7

BLACK & SCHOLES MODEL

The Black-Scholes world

The 'Perfect' (idealised) Market:

- Asset price dynamics are assumed to be GBM.
- No transaction costs.
- Continuous rebalancing of a replicating portfolio.

The B&S methodology for option pricing produces a portfolio (of the underlying stock and MM) that replicates the option payoff, just as we did earlier.

Now, however, instead of the replicating equations being linear algebraic, they are *partial differential equations* (pde) because 'time' - and portfolio rebalancing - are assumed to be continuous.

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The Black-Scholes Formula

Black & Scholes were able to solve the pde to get the price of a European call as:

$$P_c = x_0 N(d) - Ke^{-rT} N(d - \sigma\sqrt{T})$$

where:

- x_0 : the price of the underlying
- K : the strike price
- T : the time to option expiration
- r : the instantaneous risk-free interest rate
- σ : the volatility of the underlying price

In the above formula:

$$d = \frac{1}{\sigma\sqrt{T}} \left[\ln(\frac{x_0}{K}) + (r + \frac{\sigma^2}{2})T \right]$$

and $N(\cdot)$ is the Cumulative Normal distribution function.

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The Cumulative Normal Distribution

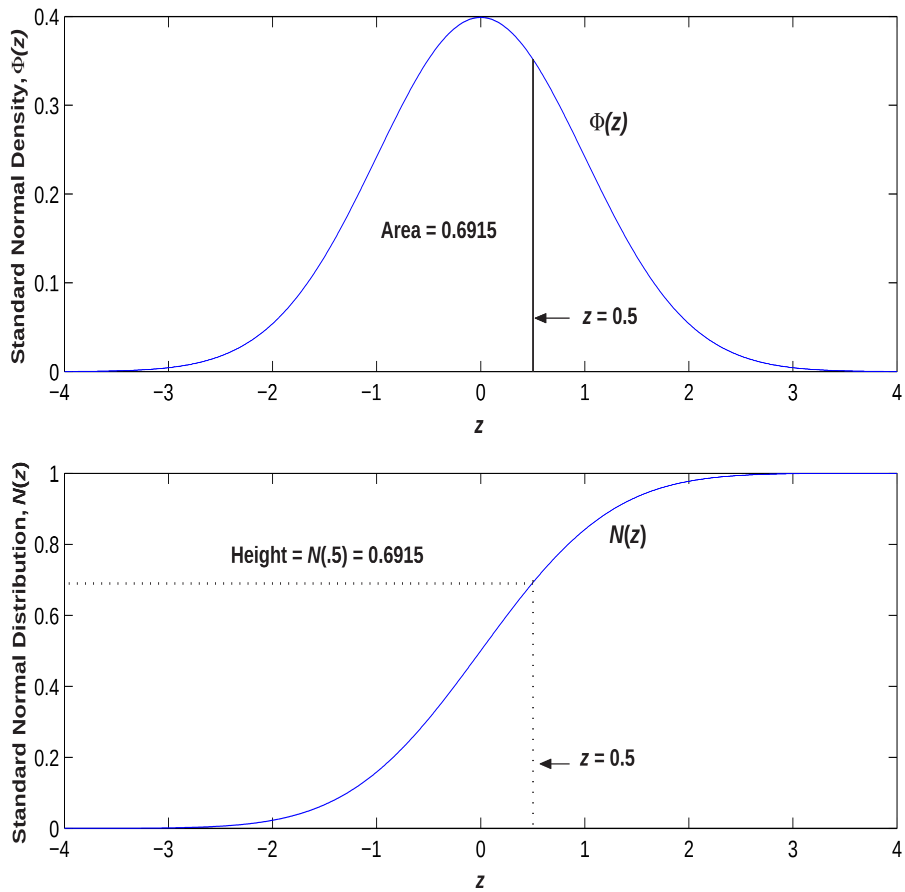

Figure 21: The Cumulative Normal and the Normal Density functions

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Put-Call Parity

Consider a call and a put both with strike price K, expiration time T, and prices P_c and P_p respectively. How are the prices related?

From purely arbitrage considerations, it is simple to show that

$$P_c - P_p = x_0 - K \cdot e^{-rT}$$

So, the B&S formula can be used to price both calls and puts.

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American options

It is never optimal to exercise an American call early.
Therefore, its price is the same as a European call and can be computed by the B&S formula.
The same is **not** true for American puts for which early exercise may be optimal and hence are worth more that a European put.

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Option price sensitivities

(Often referred to as the "Greeks")

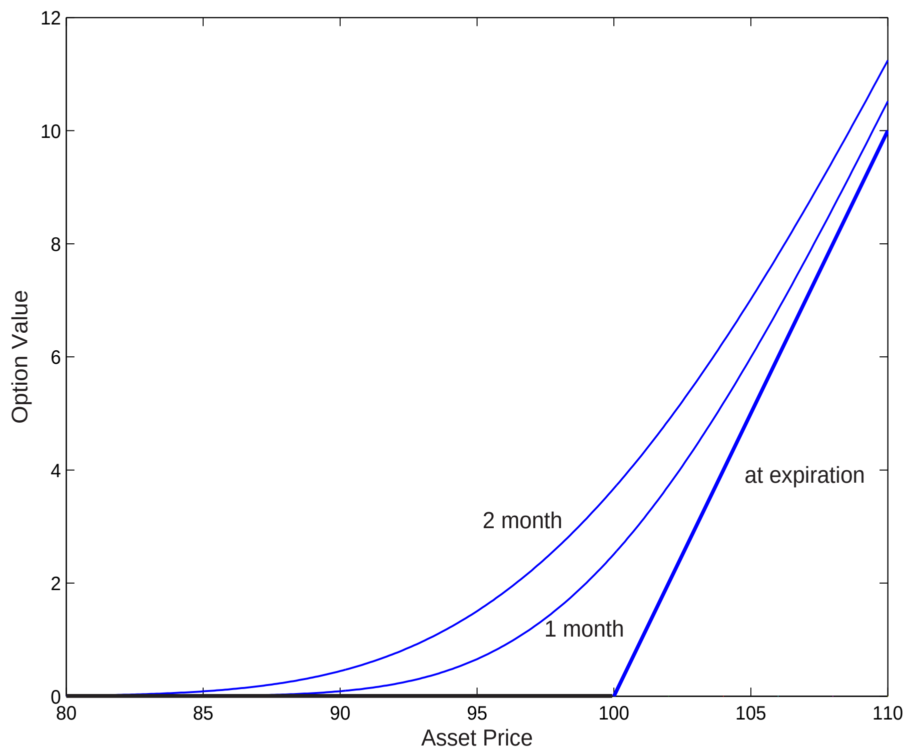

Figure 22: Option price dependence on underlying price.

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The Greeks

Rate of change of the option price w.r.t. the following:

- Delta (δ) : Underlying price
- Theta (θ) : Time to expiration
- Kappa, or Vega (κ) : Volatility
- Rho (ρ) : Interest rate

The rate of change of δ (i.e. the second derivative of option price) w.r.t. the underlying is called:

- Gamma (γ)

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The Delta of an option

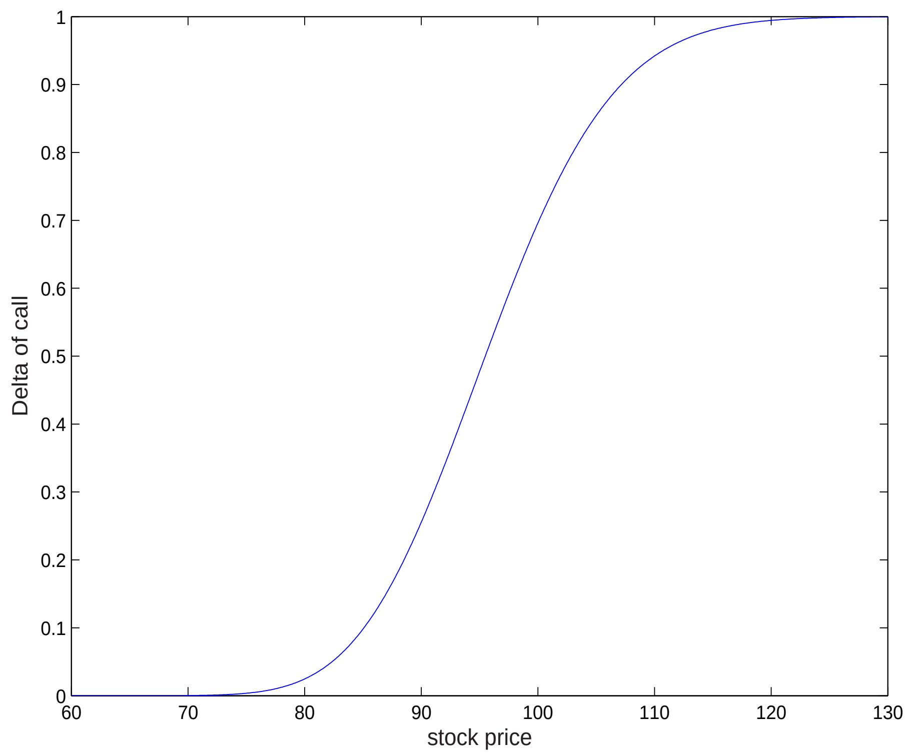

Figure 23: Delta as a function of the stock price.

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PART 8

SUMMARY, CONCLUSIONS, STATE-OF-THE-ART

1. The dynamics of many correlated assets can now be modelled by state-space graphs simultaneously. This has large implications for
   portfolio management,
   hedge funds,
   basket credit-default swaps.

2. Advanced option pricing methodologies (based on linear and non-linear optimisation) are now able to price/hedge in incomplete markets. This has implications for
   options on macroeconomic variables,
   energy options,
   mixtures of options and insurance.

3. The new pricing methodologies can deal with
   liquidity problems,
   transaction costs,
   fat-tailed return distributions,
   non-constant volatility.

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