# Robust Portfolio Optimization with Copulas

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**Abstract**

Conditional value-at-risk (CVaR) is widely used in portfolio optimization as a measure of risk. CVaR is clearly dependent on the underlying probability distribution of the portfolio. We show how copulas can be introduced to any problem that involves distributions and how they can provide solutions for the modelling of the portfolio. We use this to provide the copula formulation of the CVAR of a portfolio. Given the critical dependence of CVaR on the underlying distribution, we use a robust framework to extend our approach to Worst Case CVaR (WCVaR). WCVaR is achieved through the use of rival copulas. These rival copulas have the advantage of exploiting a variety of dependence structures, symmetric and not.

## 1 Introduction

In this paper we look into the problem of portfolio optimization where the assets of the portfolio are described by random variables. In this situation the selection of the optimal portfolio depends on the underlying assumptions on the behavior of the assets and the choice on the measure of risk. Usually the objective is to find the optimal risk-return trade-off.

One of the pioneers in portfolio optimization was Markowitz [14] who proposed the practical mean-variance framework for risk return analysis. Although the most common measure for the estimation of the return of the portfolio remains the expected return many other ways of calculating the risk have been developed. A widely used measure of risk is Value at Risk (VaR). VaR is the measure of risk that is recommended as a standard by the Basel Committee. However, VaR has been criticized in the recent years mainly for two reasons. Firstly, VaR does not satisfy sub-additivity and hence it is not a coherent measure of risk in the way that is defined by Artzner et al. [2]. As a consequence it is not a convex measure of risk and thus it may have many local extrema which cause technical issues when optimizing a portfolio. Secondly, it gives a percentile of loss distribution that does not provide an adequate picture of the possible losses in the tail of the distribution. Szego [26] uses this argument to state that ”VaR does not measure risk”. Then he suggests alternative measures of risk with one of them being Conditional Value at Risk (CVaR).

CVaR is the expectation of the distribution above VaR. Thus, the value of CVaR is affected by the fatness of the tail of the distribution. Hence, CVaR provides a better description of the loss on the tail of the distribution. Rockafellar and Uryasev [19, 20] proposed a minimization formulation that usually results in a convex or linear problem. These are desirable aspects of CVaR and have paved the way of its use in risk management and portfolio optimization. A literature review on CVaR can be found in Zhu and Fukushima [28] and the references therein.

Following the formulation of Rockafellar and Uryasev [19, 20] in order to calculate CVaR one has to make some assumptions on the uncertainty characterization of the assets. This can be in the form of an uncertainty domain like a hypercube or ellipsoidal in which all feasible uncertainty values lie. An alternative is by assuming some multivariate distribution [28]. In this paper we focus on the selection of multivariate distributions.

Gaussian distribution is the most commonly used multivariate case. It is easy to calibrate and also there are very efficient algorithms to simulate Gaussian data. This also applies to some extent to elliptical family of distributions. One disadvantage of using Gaussian distribution is its symmetry. This implies that the probability of losses is the same as the probability of gains. Studies suggest that at least in the context of financial markets, assets exhibit stronger comovements during a crisis as opposed to prosperity [1, 11, 12]. The second disadvantage is that it uses linear correlation as a measure of dependence. As the name suggests, linear correlation is characterized by linear dependencies. Since the observation of asymmetric comovements mentioned above suggests non-linear dependencies, linear correlation may not be an adequate measure of dependence [2, 26].

One way of addressing the limitations of the symmetry underlying elliptical distributions is to consider mixture distributions. A linear combination of a set of distributions is used to fit the given sample by optimizing the combination weights. Hasselblad [9, 10] was one of the first who looked into mixture distributions and how their parameters can be estimated. Zhu and Fukushima [28] avoid the assumptions needed on the set of distributions and their parameters and also avoid the estimation of the weights. Subsets of historical returns are used to represent data arising from different distributions and a worst-case scenario approach is applied to avoid the calibration of the weights. Hu [11, 12] and Smillie [25] use mixture copulas to fit their data samples but they only consider the bivariate case. The work of Hu [11, 12] and Zhu and Fukushima [28] motivates us to introduce copulas within a worst case robust scenario framework.

Copulas are multivariate distribution functions whose one-dimensional margins are uniformly distributed on the closed interval [0,1] [4, 17, 24]. The uniform margins can be replaced by univariate cumulative distributions of random variables [4, 17, 24]. This implies that copulas consider the dependency between the marginal distributions of the random variables instead of focusing directly on the dependency between the random variables themselves. This makes them more flexible than standard distributions because it is possible to separate the selection of the multivariate dependency from the selection of the univariate distribution. As an extension to that, the calibration of the multivariate distribution can be separated into two steps [4]. Also, the fact that copulas describe the dependency between the marginal distributions which are monotonic makes them invariant under monotonic transformations [4, 17]. Copulas are associated with many measures of dependence that measure the monotonic dependencies between two random variables. Furthermore, like copulas themselves these monotonic measures are invariant under monotonic transformations [7, 23, 27].

In this paper, we mainly focus on Archimedian copulas. This is a family of copulas that exhibits some interesting characteristics that can be utilized in our distribution modeling as discussed in the next section.

The paper is structured as follows. In Section 2 we introduce copulas and the associated measures of dependence together with some theoretical background. In Section 3 we derive CVaR for copulas. We extend CVaR to WCVaR through the use of mixture copulas. We conclude the section by stating the generalized optimization problem for WCVaR. In Section 4 we construct a model based on the theory of the previous sections. Then, we provide two numerical examples where we asses the performance of our model. Finally, we close with conclusions.

## 2 Copulas

Copulas arise from the theory of probabilistic metric functions and where first introduced by Sklar in 1959 [24]. Copulas are multivariate distribution functions whose one-dimensional margins are uniformly distributed on the closed interval $\mathbb{I} \equiv [0, 1]$. A more rigorous definition for copulas is given below.

**Definition 2.1.** *An n dimensional **copula** (n-copula) is a function C from $\mathbb{I}^n$ to $\mathbb{I}$ with the following properties*

1. $C(u_1, .., u_i, ..., u_n) = 0$ *if any* $u_i = 0$ *for* $i = 1, 2, ..., n$ *(we also describe a function with this property as **grounded**)*
2. $C\,(1, ..., 1, u_m < 1, ..., 1) = u_m$ *for all* $u_m \in \mathbb{I}$ *where* $m = 1, 2, ..., n$
3. $C(\mathbf{u}) \geq 0 \;\; \forall \;\; \boldsymbol{u} \in \mathbb{I}^n$ *(we also describe a function with this property as **n-increasing**)*

We continue with the definition of distribution functions and joint multivariate distribution functions. This is used when we discuss the relation between distribution functions and copulas.

**Definition 2.2.** *A distribution function is a function F from $\mathbb{R}$ to $\mathbb{I}$ with the following properties:*

1. *F is nondecreasing*
2. $F(-\infty) = 0$ *and* $F(\infty) = 1$

**Definition 2.3.** *A joint multivariate distribution function is a function from $\mathbb{R}^n$ to $\mathbb{I}$ with the following properties*

1. *F is n-increasing*
2. *F is grounded*
3. $F(\infty) = 1$
4. $F(\infty, ..., \infty, x_m, \infty, ..., \infty) = F_m(x_m)$

*where* $m = 1, 2, ..., n$*.*

We can use copulas to replace probability distributions in all of their applications thanks to Sklar's theorem. Sklar's theorem is probably the most important theorem that links copulas to probability distributions. This theorem, together with the corollary that follows provide the relation between n-Copulas and multivariate distributions. Sklar introduced his theorem in 1959 [24] where also the proof for its bivariate case can be found. The multivariate case is discussed by Schweizer and Sklar [22] (the proofs for the corollary can be found in the same references).

**Theorem 2.4 (Sklar's Theorem).** *Let F be an n-dimensional distribution function with margins $F_1, F_2, ..., F_n$. Then there exists an n-copula C such that, for all $x \in \mathbb{R}^n$,*

$$F(x_1, x_2, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)). \tag{1}$$

*Furthermore, if $F_1, ..., F_n$ are continuous, then C is unique; otherwise C is unique on $\textbf{Ran}F_1 \times ... \times \textbf{Ran}F_n$ (**Ran** ≡Range).*

**Corollary 2.5.** *Let F be an n-dimensional distribution function with margins $F_1, ..., F_n$, and let C be an n-copula. Then, for any $\boldsymbol{u} \in \mathbb{I}^n$,*

$$C(u_1, ..., u_n) = F(F_1^{(-1)}(u_1), ..., F_n^{-1}(u_n)) \tag{2}$$

*where $F_1^{-1}, ..., F_n^{-1}$ are the quasi-inverses of the marginals.*

The margins $F_1, ..., F_n$ and the multivariate distribution function F are as defined by Definitions 2.2 and 2.3. The reason that $u_i$ can be replaced by $F_i(x_i)$ is because they both belong to the domain $\mathbb{I}$ and they both are uniformly distributed (Let $u \sim U(0, 1)$ then $P(F(x) \leq u) = P(x \leq F^{-1}(u)) = F(F^{-1}(u)) = u$).

Using Theorem 2.4 and Corollary 2.5 we can also derive the relation between the probability density functions and the copulas. In the following definition $f$ is the multivariate probability density function of the probability distribution $F$ and $f_1, ..., f_n$ are the univariate probability density functions of the margins $F_1, ..., F_n$.

**Definition 2.6.** *The **copulas density** of a n-copula C is the function $c : \mathbb{I}^n \to [0, \infty)$ such that*

$$c(u_1, ..., u_n) \equiv \frac{\partial^n C(u_1..., u_n)}{\partial u_1...\partial u_n} = \frac{f(x_1, ..., x_n)}{\prod_{i=1}^n f_i(x_i)} \tag{3}$$

We can see from (1)-(3) that copulas decompose the multivariate probability distribution from its margins. The margins $F_1, ..., F_n$ can be any distribution of our choice while the copula simply describes the monotonic relation between the margins. This is one of the biggest advantages of copulas because they separate the problem of finding the correct distribution into two parts, one is finding the distribution of the margins and second the dependency between them. This is much easier from finding directly the multivariate dependency between the random variables. Hence, the calibration of the copulas become an easier task. The calibration methods for copulas can be found in Cherubini et al. [4]. Also, an introduction to copulas can be found in Nelsen [17] and Schweizer and Sklar [22] discuss the relationship of copulas to probabilistic metric spaces and the underlying theory.

## 2.1 Special cases of copulas and related measures of dependence

We introduce the copulas to be used in our examples. Together with the copulas we consider their associated measures of dependence. The focus is on a special family of copulas called *Archimedian*. We also consider the Gaussian copula which is the copula version of the multivariate normal distribution.

Archimedian copulas were firstly introduced by Ling [13]. They belong to the family of probabilistic metric spaces that have some of the properties of Archimedes triangle function and hence the name [22]. This is a family of copulas that arises differently from the rest. Instead of using Theorem 2.4 we construct them using directly a function $\varphi$, known as a generator, which enables us to write the expression for the copula in a closed form.

**Definition 2.7.** *Given a function $\varphi : \mathbb{I} \to [0, \infty)$ such that $\varphi(1) = 0$ and $\varphi(0) = \infty$ and having inverse $\varphi^{(-1)}$ completely monotone, an n-place **Archimedean** copula is a function $C_\varphi : \mathbb{I}^n \to \mathbb{I}$ such that*

$$C_\varphi(\boldsymbol{u}) = \varphi^{(-1)}(\varphi(u_1) + ... + \varphi(u_n)).$$

An extended literature regarding the Archimedean copulas can be found in Cherubini et al. [4], Nelsen [17], Schweizer and Sklar [22]. In our analysis we focus on three Archimedean copulas, *Clayton, Gumbel* and *Frank*. Our motivation for using these particular copulas comes from Hu [11, 12]. Hu [11, 12] focus is the calibration of bivariate mixture copula (see also Hasselblad [9, 10] for mixture distributions). The reasons that he chose these particular copulas are mainly two. Each copula better describes a different type of dependency. Clayton and Gumbel are non symmetric copulas that describe more adequately positive and negative dependencies respectively. Frank copula is symmetric but it has different properties to Gaussian copula. Hence, by using them in a mixture structure we cover a large spectrum of possible dependencies. Also, these three copulas are very easy to calibrate.

The definitions of the three Archimedean copulas are the following:

**Definition 2.8.** *Given a generator of the form $\varphi(u) = u^{-\alpha} - 1$ with $\alpha \in (0, \infty)$ then, the **Clayton** n-copula is given by*

$$C_{Cl}(\boldsymbol{u}) = \max[(u_1^{-\alpha} + ... + u_n^{-\alpha} - n + 1)^{-1/\alpha}, 0].$$

**Definition 2.9.** *Given a generator of the form $\varphi(u) = (-\ln(u))^\alpha$ with $\alpha \in (1, \infty)$ then, the **Gumbel** n-copula is given by*

$$C_{Gu}(\boldsymbol{u}) = \exp\Big\{ - [(-\ln u_1)^\alpha + ... + (-\ln u_n)^\alpha]^{1/\alpha} \Big\}.$$

**Definition 2.10.** *Given a generator of the form $\varphi(u) = \ln\left(\frac{exp(-\alpha u)-1}{exp(-\alpha)-1}\right)$ with $\alpha \in (0, \infty)$ then, the **Frank** n-copula is given by*

$$C_{Fr}(\boldsymbol{u}) = -\frac{1}{\alpha} \ln\left\{1 + \frac{(e^{-\alpha u_1} - 1) \cdot ... \cdot (e^{-\alpha u_n} - 1)}{(e^{-\alpha} - 1)^{n-1}}\right\}.$$

For the calibration of the free parameters $\alpha$ of the Archimedean copulas we will use Kendal's $\tau$. Kendal's $\tau$ is a bivariate measure of dependence and is defined by the following equation

$$\begin{aligned} \tau(X_1, X_2) &= 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x_1, x_2) dF(x_1, x_2) - 1 \\ &= 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1. \end{aligned} \tag{4}$$

As we can see from equation (4), Kendal's $\tau$ measures the dependency between the cumulative distributions of random variable $X_1$ and $X_2$ and does not depend on the random variables themselves. Thus, $\tau$ is a measure of monotonic dependence and is invariant under monotonic transformations. This makes it a more robust measure of dependence when compared to linear correlation. For comparison purposes, we also define linear correlation as

$$\begin{aligned} \varrho(X_1, X_2) &= \frac{1}{\sigma(X_1), \sigma(X_2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F(x_1, x_2) - F_1(x_1) F_2(x_2)] dx_1 dx_2 \\ &= \frac{1}{\sigma(X_1), \sigma(X_2)} \int_0^1 \int_0^1 [C(u_1.u_2) - uv] dF_1^{(-1)}(u_1) dF_2^{(-1)}(u_2) \end{aligned} \tag{5}$$

where $\sigma(X_i)$ denotes the variance of the random variable $X_i$. It can be seen that in the copula version of of $\varrho$, the dependency on the random variables $X_1$ and $X_2$ remains in in form of the volatility $\sigma(X_i)$. An extensive literature on monotonic measures of dependence can be found in Nelsen [17], Schweizer and Wolff [23], Wolff [27] and the references therein.

Let us denote the free parameter $\alpha$ of each of the Archimedian copulas by $\alpha_{Cl}$ for $C_{Cl}$ (Definition 2.8), $\alpha_{Gu}$ for $C_{Gu}$ (Definition 2.9) and $\alpha_{Fr}$ for $C_{Fr}$ (Definition 2.10). For these three cases we have closed form relations with Kendall's $\tau$ equation (4) [4]. For $C_{Cl}$ we have that

$$\tau = 1 - \alpha_{Cl}^{-1}, \tag{6}$$

for $C_{Gu}$

$$\tau = \frac{\alpha_{Gu}}{\alpha_{Gu} + 2}, \tag{7}$$

and for $C_{Fr}$ we have

$$\tau \quad = \quad 1 + \frac{4[D_1(\alpha_{Fr})]}{\alpha_{Fr}} \tag{8}$$

$$\tag{9}$$

where

$$D_k(\alpha) = \frac{k}{\alpha^k} \int_0^{\alpha} \frac{x}{\exp(x) - 1} dx \ \text{ for } \ k = 1, 2.$$

As we can see from Definitions 2.8-2.10 there is only one free parameter to calibrate regardless of the dimensions of the copula. On the other hand we have a $\tau$ for each pair of random variables. Our solution to this problem is to calculate the $\tau$ for all the pairs and select the largest. This seems to be consistent with the use of copulas within a worst case robust framework as it provides the extreme scenario for each copula.

We define the Gaussian copula as it is the most commonly used by practitioners [4, 5, 7, 21, 25].

**Definition 2.11.** *Given a n-place standard multivariate normal distribution function $\Phi_n$ parameterized by a dispersion matrix $P \in [-1, 1]^{n \times n}$, the* ***Gaussian copula*** *is the function $C_{Ga} : \mathbb{I}^n \to \mathbb{I}$ such that*

$$C_{Ga}(\boldsymbol{u}) = \Phi(\Phi_1^{-1}(u_1), ..., \Phi_n^{-1}(u_n)). \tag{10}$$

For $C_{Ga}$ to be called Gaussian copula all the margins $\{\Phi_i\}_{i=1}^n$ have to be normally distributed but they can have different mean and variance.

## 3 Worst Case Value at Risk

Having introduced the theorems that enable us to associate copulas with distributions we will derive the copula formulation of *Worst Case Value at Risk* (WCVaR). In order to show how WCVaR can be derived for copulas we follow the same approach as Zhu and Fukushima [28]. At every step involving the use of distributions we present the equivalent copula formulation. For the derivation of the copula formulation we use equations (1)-(3).

In order to define the WCVaR we first have to define *Value at Risk* (VaR) and *Conditional Value at Risk* (CVaR). We first give VaR

**Proposition 3.1.** *Let* $\mathbf{w} \in \mathbb{W} \subseteq \mathbb{R}^m$ *be decision vector,* $\mathbf{u} \in \mathbb{I}^n$ *be a random vector,* $\tilde{g}(\mathbf{w}, \mathbf{u})$ *the cost function and* $\mathbf{F}(\mathbf{x}) = (F_1(x_1), ..., F_n(x_n))$ *a set of marginal distributions where* $\mathbf{u} = \mathbf{F}(\mathbf{x})$*. Also, let assume that* $\mathbf{u}$ *follows a continuous distribution with copula density function* $c(.)$*. Then* $VaR_\beta$ *for a confidence level* $\beta$ *is defined as*

$$VaR_\beta(\mathbf{w}) \triangleq \min\{\alpha \in \mathbb{R} : C(\mathbf{u}|\tilde{g}(\mathbf{w}, \mathbf{u}) \leq \alpha) \geq \beta\}. \tag{11}$$

Given a decision $\mathbf{w} \in \mathbb{W}$ and a random vector $\mathbf{x} \in \mathbb{R}^n$ which follows a continuous distribution with density function $f(.)$, the probability of $g(\mathbf{w}, \mathbf{x})$ not exceeding a threshold $\alpha$ is represented as

$$\begin{aligned}
\Psi(\mathbf{w}, \alpha) &\triangleq \int_{g(\mathbf{w},\mathbf{x})\leq\alpha} f(\mathbf{x})d\mathbf{x} \\
&\triangleq \int_{g(\mathbf{w},\mathbf{x})\leq\alpha} c(\mathbf{F}(\mathbf{x})) \prod_{i=1}^{n} f_i(x_i)d\mathbf{x} \\
&\triangleq \int_{\tilde{g}(\mathbf{w},\mathbf{u})\leq\alpha} c(\mathbf{u})d\mathbf{u} \\
&\triangleq C(\mathbf{u}|\tilde{g}(\mathbf{w}, \mathbf{u}) \leq \alpha),
\end{aligned}$$

where $f_i(x_i) = \frac{\partial F_i(x_i)}{\partial x_i}$ is the univariate probability distribution of the individual elements of the random vector $\mathbf{x}$ (see Definition 2.6). $\tilde{g}(\mathbf{w}, \mathbf{u}) = g(\mathbf{w}, \mathbf{F}^{-1}(\mathbf{u}))$ where $\mathbf{F}^{-1}(\mathbf{u}) = (F_1^{-1}(u_1), ..., F_m^{-1}(u_n))$ maps the domain of the cost function from $\mathbb{R}^n$ to $\mathbb{I}^n$, as implied by the transformation $u_i = F_i(x_i)$. For the deriviation of the copula version of $\Psi(\mathbf{w}, \alpha)$ we use equation (3). Having defined $\Psi(\mathbf{w}, \alpha)$, we consider the VaR. Given a fixed $\mathbf{w} \in \mathbb{W}$ and a confidence level $\beta$, VaR is defined as

$$\begin{aligned}
VaR_\beta(\mathbf{w}) &\triangleq \min\{\alpha \in \mathbb{R} : \Psi(\mathbf{w}, \alpha) \geq \beta\} \\
&\triangleq \min\{\alpha \in \mathbb{R} : C(\mathbf{u}|\tilde{g}(\mathbf{w}, \mathbf{u}) \leq \alpha) \geq \beta\}
\end{aligned}$$

which give as equation (11). We continue with the definition of the CVaR with respect to VaR.

**Proposition 3.2.** *Given* $\mathbf{w}$*,* $\mathbf{u}$*,* $\mathbf{F}(\mathbf{x})$ *and* $\tilde{g}(\mathbf{w}, \mathbf{u})$ *as in Proposition 3.1 we define* $CVaR_\beta$ *for a confidence level* $\beta$ *as*

$$CVaR_\beta(\mathbf{w}) \triangleq \frac{1}{1-\beta} \int_{\tilde{g}(\mathbf{w},\mathbf{u})\geq VaR_\beta(\mathbf{w})} \tilde{g}(\mathbf{w}, \mathbf{u})c(\mathbf{u})d\mathbf{u}. \tag{12}$$

Again, we start from the equation of CVaR that arise from the probability density function $f(.)$ and we derive the copula form.

$$\begin{aligned} CVaR_\beta(\mathbf{w}) &\triangleq \frac{1}{1-\beta}\int_{g(\mathbf{w},\mathbf{x})\geq VaR_\beta(\mathbf{w})} g(\mathbf{w},\mathbf{x})f(\mathbf{x})d\mathbf{x} \\ &\triangleq \frac{1}{1-\beta}\int_{g(\mathbf{w},\mathbf{x})\geq VaR_\beta(\mathbf{w})} g(\mathbf{w},\mathbf{x})c(\mathbf{F}(\mathbf{x}))\prod_{i=1}^{n} f_i(x_i)d\mathbf{x} \\ &\triangleq \frac{1}{1-\beta}\int_{\tilde{g}(\mathbf{w},\mathbf{u})\geq VaR_\beta(\mathbf{w})} \tilde{g}(\mathbf{w},\mathbf{u})c(\mathbf{u})d\mathbf{u} \end{aligned} \tag{13}$$

which gives as CVaR as defined in Proposition 3.2. Following Rockafellar and Uryasev [19] we formulate equation (13) as the following minimization problem

$$\begin{aligned} G_\beta(\mathbf{w},\alpha) &\triangleq \alpha + \frac{1}{1-\beta}\int_{\mathbf{x}\in\mathbb{R}^n} [g(\mathbf{w},\mathbf{x})-\alpha]^+ f(\mathbf{x})d\mathbf{x} \\ &\triangleq \alpha + \frac{1}{1-\beta}\int_{\mathbf{u}\in\mathbb{I}^n} [\tilde{g}(\mathbf{w},\mathbf{u})-\alpha]^+ c(\mathbf{u})d\mathbf{u}. \end{aligned} \tag{14}$$

Hence, we have

$$CVaR_\beta(\mathbf{x}) = \min_{\alpha\in\mathbb{R}} G_\beta(\mathbf{w},\alpha). \tag{15}$$

By solving the minimization problem in equation (15), we directly obtain both the values of CVaR and VaR. From Proposition 3.1 we have that the value of VaR is the value of $\alpha$.

In order for the above definitions to be computed, exact knowledge of the distribution $f(\mathbf{x})$ or copula density $c(\mathbf{u})$ and the margins $\mathbf{F}(\mathbf{x})$ is needed. As the aim in this paper is to represent distributions with copulas, we shall omit using $f(\mathbf{x})$ and use $c(\mathbf{u})$ instead. The equivalence of the two is discussed in the previous section. Knowledge of the copula $C(\mathbf{u})$ and its margins $\{u_i = F_i(x_i)\}_{i=1}^n$ implies knowledge of $f(\mathbf{x})$ and $c(\mathbf{u})$. A copula representation of the distribution of $\mathbf{x}$ cannot be expected to be exact. Thus, we assume that our copula representation belongs to a set of copulas $c(.) \in \mathscr{C}$. In order to be robust, from that set we want to choose the worst performing copula or copulas, as the worst-case might not be unique. Hence, we define WCVaR.

**Definition 3.3.** *The Worst-case CVaR (WCVaR) for fixed $\mathbf{w} \in \mathbb{W}$ with respect to $\mathscr{C}$ is defined as*

$$WCVaR_\beta(\mathbf{w}) \triangleq \sup_{c(.)\in\mathscr{C}} CVaR_\beta(\mathbf{w}). \tag{16}$$

Is known that CVaR is a coherent measure of risk [2, 26, 28]. For a measure of risk $\rho$ mapping a random vector $X$ to be coherent it has to satisfy the following properties:

(i) Subadditivitty: for all random vectors $X$ and $Y$, $\rho(X+Y) \leq \rho(X) + \rho(Y)$;

(ii) Positive homogeneity : for positive constant $\lambda$, $\rho(\lambda X) = \lambda\rho(X)$;

(iii) Monotonicity: if $X \leq Y$ for each outcome, then $\rho(X) \leq \rho(Y)$;

(iv) Translation invariance: for constant $m$, $\rho(X+m) = \rho(X) + m$.

Zhu and Fukushima [28] prove that WCVaR preserves coherence. They also give the following lemma from Fan [8] which allows to formulate the problem into a tractable one.

**Lemma 3.4.** *Suppose that* $\mathbb{W}$ *and* $\mathcal{X}$ *are nonempty convex sets in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$*, respectively, and the function* $g(\mathbf{w},\mathbf{x})$ *is convex in* $\mathbf{w}$ *for any* $\mathbf{x}$*, and concave in* $\mathbf{x}$ *for any* $\mathbf{w}$*. Then we have*

$$\min_{\mathbf{w}\in\mathbb{W}}\max_{\mathbf{x}\in\mathbf{X}} g(\mathbf{w},\mathbf{x}) = \max_{\mathbf{x}\in\mathbf{X}}\min_{\mathbf{w}\in\mathbb{W}} g(\mathbf{w},\mathbf{x}) \tag{17}$$

We also use Lemma 3.4 to extend the proof from Zhu and Fukushima [28] to copulas and eventually formulate our problem as a *minmax* problem.

## 3.1 Mixture Copula

In this example the distribution of the vector of returns $\mathbf{x}$ is described by a mixture copula

$$C(\mathbf{F}(\mathbf{x})) = \lambda^T\vec{C}, \tag{18}$$

where $\lambda \in \Lambda = \{\lambda : \mathbf{e}^T\lambda = 1, \lambda \geq \mathbf{0}, \lambda \in \mathbb{R}^l\}$ and $\vec{C} = (C_1(\mathbf{F}(\mathbf{x})), ...C_l(\mathbf{F}(\mathbf{x})))$ is the vector with copulas and $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), ..., F_1(\mathbf{x}))$ is the vector of the cumulative univariate distributions. We can apply equation (3) to equation (18) to obtain the density of the mixture copula. Then, we can use this density in the equations in Section 3.1 to obtain

$$G_\beta(\mathbf{w},\alpha) = \alpha + \frac{1}{1-\beta}\int_{\mathbf{u}\in\mathbb{I}^n}[\tilde{g}(\mathbf{w},\mathbf{u})-\alpha]^+\sum_{i=1}^{l}\lambda_i c_i(\mathbf{u})d\mathbf{u} = \sum_{i=1}^{l}\lambda_i G_\beta^i(\mathbf{w},\alpha), \tag{19}$$

where

$$G_\beta^i(\mathbf{w},\alpha) = \alpha + \frac{1}{1-\beta}\int_{\mathbf{u}\in\mathbb{I}^n}[\tilde{g}(\mathbf{w},\mathbf{u})-\alpha]^+ c_i(\mathbf{u})d\mathbf{u} \quad \text{for} \quad i = 1,2,...,l. \tag{20}$$

Then, the optimization problem that we need to solve is stated by the following theorem and corollary from Zhu and Fukushima [28]:

**Theorem 3.5.** *For each* $\mathbf{w}$*,* $WCVaR_\beta(\mathbf{w})$ *with respect to* $\mathscr{C}$ *is given by*

$$WCVaR_\beta(\mathbf{w}) = \min_{\alpha\in\mathbb{R}}\max_{\lambda\in\Lambda} G_\beta(\mathbf{w},\alpha), \tag{21}$$

*where* $\Lambda = \{\lambda : \mathbf{e}^T\lambda = 1, \lambda \geq \mathbf{0}, \lambda \in \mathbb{R}^l\}$.

**Corollary 3.6.** *Minimizing* $WCVaR_\beta(\mathbf{w})$ *over* $\mathbb{W}$ *can by the following minimization*

$$\min_{\mathbf{w}\in\mathbb{W}} WCVaR_\beta(\mathbf{w}) = \min_{\mathbf{w}\in\mathbb{W}}\min_{\alpha\in\mathbb{R}}\max_{\lambda\in\Lambda} G_\beta(\mathbf{w},\alpha). \tag{22}$$

*More specifically, if* $(\mathbf{w}^*,\alpha^*,\lambda^*)$ *attains the right hand side minimum, then* $(\mathbf{w}^*$ *attains the left-hand side minimum.*

The proofs of the Theorem 3.5 and Corollary 3.6 can be found in Zhu and Fukushima [28]. They provide the proof for the case of mixture distributions. The theorems in Section 2 together with Proposition 3.1 and Proposition 3.2 show that Theorem 3.5 and Corollary 3.6 can be applied to copulas. For the sake of completeness we give the logic behind the proof and we continue with the formulation of the optimization problem.

In order to optimize the portfolio we need to solve

$$\min_{\mathbf{w}\in\mathbb{W}} WCVaR_\beta(\mathbf{w}) \equiv \min_{\mathbf{w}\in\mathbb{W}} \max_{\lambda\in\Lambda} \min_{\alpha\in\mathbb{R}} G_\beta(\mathbf{w},\alpha) \tag{23}$$

Since the mixture copula (18) is linear in $\lambda$, Zhu and Fukushima [28] use Lemma 3.4 to show that (23) can be written as

$$\min_{\mathbf{w}\in\mathbb{W}} \min_{\alpha\in\mathbb{R}} \max_{\lambda\in\Lambda} G_\beta(\mathbf{w},\alpha). \tag{24}$$

Then, Zhu and Fukushima [28] extend equation (24) one step further by using equation (19). The equivalent to equation (24) is

$$\min_{(w,\alpha,\theta)\in\mathbb{W}\times\mathbb{R}\times\mathbb{R}} \left\{ \theta : \sum_{i=1}^{l} \lambda_i G_\beta^i(\mathbf{w},\alpha) \leq \theta, \forall\lambda \in \Lambda \right\} \tag{25}$$

and $\theta$ must satisfy

$$G_\beta^i(\mathbf{w},\alpha) \leq \theta, \quad \text{for} \quad i = 1,2,...,l. \tag{26}$$

Equation (25) can thus be reduced to

$$\min_{(w,\alpha,\theta)\in\mathbb{W}\times\mathbb{R}\times\mathbb{R}} \left\{ \theta : G_\beta^i(\mathbf{w},\alpha) \leq \theta, i = 1,2,...,l. \right\} \tag{27}$$

A straightforward approach to evaluating equation (27) is by Monte Carlo simulation. Rockafellar and Uryasev [19] give an approximation of $G_\beta(\mathbf{w},\alpha)$,where Monte Carlo simulation can be used. They write $G_\beta(\mathbf{w},\alpha)$ as

$$\hat{G}_\beta(\mathbf{w},\alpha) = \alpha + \frac{1}{S(1-\beta)} \sum_{k=1}^{S} [\tilde{g}(\mathbf{w},\mathbf{u}_{[k]}) - \alpha]^+. \tag{28}$$

where $\mathbf{u}_{[k]}$ is the $k^{th}$ sample vector (again here we give the copula version where $\mathbf{u}_{[\mathbf{k}]} = \mathbf{F}(\mathbf{x}_{[k]})$). Thus, using equation (28) we can express equation (27) for evaluation using Monte Carlo simulations

$$\min_{(w,\alpha,\theta)\in\mathbb{W}\times\mathbb{R}\times\mathbb{R}} \left\{ \theta : \alpha + \frac{1}{S^i(1-\beta)} \sum_{k=1}^{S^i} [\tilde{g}(\mathbf{w},\mathbf{u^i}_{[k]}) - \alpha]^+ \leq \theta, i = 1,2,...,l. \right\}, \tag{29}$$

where $\mathbf{u^i}_{[k]}$ is the $k^{th}$ sample arising from copula $C_i$ of the mixture copula (equation (18)). $S^i$ is the size of the sample that arises from $C_i$.

Following Zhu and Fukushima [28] we write the minimization problem as

$$\min \quad \theta \tag{30}$$

$$\text{s.t.} \quad \mathbf{w} \in \mathbb{W}, \mathbf{v} \in \mathbb{R}^m, \alpha \in \mathbb{R}, \theta \in \mathbb{R} \tag{31}$$

$$\alpha + \frac{1}{S^i(1-\beta)}(\mathbf{1}^i)^T \mathbf{v}^i \leq \theta,\ i = 1,...,l, \tag{32}$$

$$v_k^i \geq \tilde{g}(\mathbf{w},\mathbf{u^i}_{[k]}) - \alpha,\ k = 1,...,S^i,\ i = 1,...,l, \tag{33}$$

$$v_k^i \geq 0,\ k = 1,...,S^i,\ i = 1,...,l,, \tag{34}$$

where $\mathbf{v} = (\mathbf{v}^1;...;\mathbf{v}^l) \in \mathbb{R}^m$ with $m = \sum_{i=1}^{l} S^i$ and $\mathbf{1}^i = (1;..;1) \in \mathbb{R}^{S^i}$.

# 4 Portfolio management under a worst case copula scenario

In this section we demonstrate how the theory in Section 3 can be used for the optimization of a portfolio of financial assets. Financial assets can be described by distributions and subsequently their risk can be measured with the use of CVaR.

We consider a portfolio of $n$ financial assets $A_1, ...A_n$. We assume that the returns of the assets are log-normally distributed [3]

$$x_i = \frac{dA_i(t)}{A_i(t)} = \mu_i dt + \sigma_i dB_i(t), \tag{35}$$

where $\mu_i$ and $\sigma_i$ is the mean and the variance of the random variable $x_i$ and $dB_i(t)$ denotes a Wiener process. Hence, we have the return vector $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\mathbf{u} = (u_1, ..., u_n) = (\Phi_1(x_1), ..., \Phi_n(x_n))$. We also define the decision vector $\mathbf{w} = (w_1, ..., w_n) \in \mathbb{R}^n$, which denotes the amount of investment that we have in each financial asset in our portfolio. We also define the loss function

$$\tilde{g}(\mathbf{w}, \mathbf{u}) = -\mathbf{w}^T \mathbf{\Phi}^{-1}(\mathbf{u}), \tag{36}$$

where $\mathbf{\Phi}^{-1}(\mathbf{u}) = (\Phi_1^{-1}(u_1), ..., \Phi_n^{-1}(u_n))$. Hence, the loss function is the negative of the portfolio return $\mathbf{w}^T \mathbf{\Phi}^{-1}(\mathbf{u})$.

Having chosen the univariate distributions for the asset returns, we now consider the selection of the copula that describes the dependency between these returns. We first solve a simple optimization problem using the Gaussian copula. Consider the problem

$$\min_{\mathbf{w} \in \mathbb{W}} CVaR(\mathbf{w}), \tag{37}$$

where $\mathbb{W}$ defines the domain of $\mathbf{w}$ as described by its constrains and $\mathbf{u} \sim C_{Ga}$ (see equation (13)). This is equivalent to (30)-(34) when $l = 1$.

The advantage of problem (37) is that it employs the Gaussian copula. The latter is the most commonly used copula in practice for characterising multivariate dependencies. It is also easy to use. Furthermore, the Gaussian copula is a desirable reference point for assessing portfolio performance. There are, however drawbacks to the Gaussian copula. The first is its symmetry. Studies show that assets have stronger negative comovements than positive [1, 11, 12]. A second disadvantage is the linear correlation that can only capture linear dependencies between assets. This may not be adequate [2, 26]. Both of these disadvantages may lead to bad performance in the presence of market shocks.

By using a mixture copula we aim to conmpesate for some of these disadvantages. In expression (38), the set $\mathscr{C}$ contains the copulas from Section 2.1. The aim is to provide cover for all the types of dependencies and thereby use a robust measure of dependence (Kendall's $\tau$ (4)). This robustness is further augmented by the worst-case approach. Thus, the second problem we solve is

$$\min_{\mathbf{w} \in \mathbb{W}} WCVaR(\mathbf{w}), \tag{38}$$

where $\mathbb{W}$ is defined as above and $\mathbf{u} \sim c$ and $c \in \mathscr{C}$ is a set of copulas (see equation (16))

Problems (37) and (38) are solved using equations (30)-(34). We assume $\mathbb{W}$ to be convex and, without loss of generality (and for simplicity) we define it with the following constraints:

$$\mathbf{e}^T \mathbf{w} = 1. \tag{39}$$

Furthermore, to assure portfolio diversification, the additional constrains

$$\underline{\mathbf{w}} \leq \mathbf{w} \leq \overline{\mathbf{w}} \tag{40}$$

can be imposed where $\underline{\mathbf{w}}, \overline{\mathbf{w}} \in [-1, 1]$ are lower and upper bounds respectively.

Finally, since we optimize an asset portfolio we are interested in its performance. Hence, it is often desirable to impose an additional performance restrictions in terms of the minimum expected return $\mu$

$$E(\mathbf{w}^T \mathbf{F}^{-1}(\mathbf{u})) \geq \mu. \tag{41}$$

## 4.1 Numerical examples

We use the following seven indices: Nikkey225, FTSE100, Nasdaq, DAX30, Sensex, Bovespa, Gold index. These represent six different stock exchange markets from different parts of the world and one commodity index. The markets corresponding to the indices are Japan, England, USA, Germany, India and Brazil. These markets, with the inclusion of the commodity are intened to lead to a diversified portfolio.

The data used covers the period November 1998 - July 2011. This time line includes the dot-com bubble, South American crisis and Asian crisis. This three events took place between 1998 and 2002 and they had a large negative impact on the world markets. The data also include the 2008 Global Recession crisis. Both periodes of crises can be observed in Figure 1.

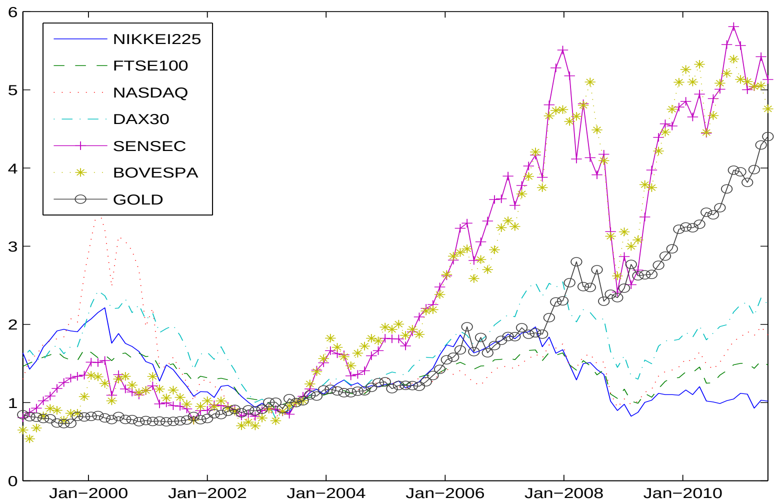


Figure 1: Seven indexed from 1998 to 2011. All the indices are normalized to 1 on July 2003

In both problems that we solve (see equations (37) and (38)), we use the period between 1998 and 2003 (1200 time steps) to calibrate the models during a crisis. We expect to show that the risk and the dependencies between assets during a crisis period can be assesed more efficiently with the use of WCVaR as opposed to using only Gaussian copula CVaR. The Worst Case Portfolio (WCP), given by (38) should perform more robustly than the Gaussian Portfolio (GP), given by (37). Overall, and particularly during a period of crisis, we expect that the WCP will perform better in the presence of downside shocks. Thus, the biggest test to the performance of the two portfolios will be the 2008 sub-mortgage crisis.

#### 4.1.1 Static portfolio

In our first example, we consider a static portfolio in which the weights of the portfolio are calculated only once. Rebalancing is done using the same weights throughout the entire lifespan of the portfolio. For the computatuion of the weights we calibrate our copulas using the period between 1998 to 2003 and then solve equations (37) and (38). The means and the variances of the seven assets as estimated from the period between 1998 to 2002 are given in Table 1.

The parameters in Table 1 are used for the univariate distributions of the assets as defined by equation (35). The univariate distributions together with the four copulas of Section 3 are used to run Monte Carlo simulations in order to provide the inputs $\mathbf{u}^i_{[k]}$ needed to solve equations (30) - (34).

Table 1: The mean and variance of the seven assets between November 1998 and June 2003

| $(10^{-3})$ | Mean | Variance |
|---|---|---|
| Nikkey225 | -0.41 | 0.21 |
| FTSE100 | -0.32 | 0.12 |
| Nasdaq | -0.22 | 0.45 |
| DAX30 | -0.38 | 0.23 |
| Sensex | 0.18 | 0.31 |
| Bovespa | 0.36 | 1.14 |
| Gold | 0.14 | 0.06 |

Simulating data from Gaussian copula is straight forward with the use of the Cholesky decomposition of the correlation matrix. This, together with the ease of calibration is what makes the Gaussian copula attractive. While simulating data from Gaussian copula is straight forward, simulating data from other copulas in general can be a difficult task which makes them less attractive. To simulate data from the three Archimedian copulas we use the algorithms found in Melchiori [16]. Melchiori [16] provides a summary of the results from Devroye [6], Marshall and Olkin [15], Nolan [18].

To solve equations (30)-(34) we also have to define the constrains $\mathbb{W}$. We use as a starting point Equations (39)-(41). We specify $\mathbf{w} \geq 0$. This implies that we only allow long positions in the portfolio. The upper bound $\overline{\mathbf{w}}$ in equation (40) is implicitly implied by equation (39).

The optimization problems are solved using the Yalmip Matlab package, together with the CPLEX solver. The PC used for the implementation of our numerical examples is a Intel Core 2 Duo, 2.8GHz with a 4 GB memory. For each of the four copulas we run 10000 simulations for each of the seven assets. Both simulations and solving the two problems (37) and (38) take less than a minute.

The two problems (37) and (38) are solved for $\mu = 0$ to 0.00025. The results are presented in Figure 2 and Table 2. In Table 2 we show the details for the performance of the two portfolios (GP and WCP) both for 'In Sample' and 'Out of Sample'. 'In Sample' performance shows that the lower bound $\mu$ is always satisfied at least by one of the two portfolios. Also, the overall (in and out of sample) performance of the WCP portfolio has always higher volatility and CVaR. This is expected since the WCP has to satisfy more constraints in the optimization problem i.e. the same contraints that exist for

Gaussian copula in the GP has to be satisfied for all four copulas used in the WCP. Hence, the CVaR obtained for the WCP is the CVaR of the worst case copula, which is the equivalent of $\theta$ requirement in equation (26). This copula also has fat tails. Hence, the higher volatility.

A WCP would be expected to perform better than the GP at least under worst case scenarios. This should apply throughout the out of sample testing period. Hence, we focus on the 'Out of Sample' period. In Figure 2 it can be verified that the performance of GP and WCP are very similar up to 2008. When the 2008 crisis occurs, the WCP outperforms the GP and maintains this advantahe thereafter. This can be seen even more clearly from the lower Figure 2 where we show the difference between the two portfolios. The difference starts to significantly increase during 2008-2011.

In the case of of $\mu = 0$ the robust portfolio outperforms the Gaussian portfolio in the Average Return (AR), Total Return (TR) and Maximum Drawdowns (MD1 and MD2). The advantage of WCP on the AR and TR remains while $\mu < 0.00015$. Although we do not get a better return form the WCP when $\mu \geq 0.0002$, in Table 1 we see that the MD2 statistics are always better for the the WCP, i.e. the losses of the WCP during the 2008 crisis where smaller than the GP and the WCP performed more robustly.

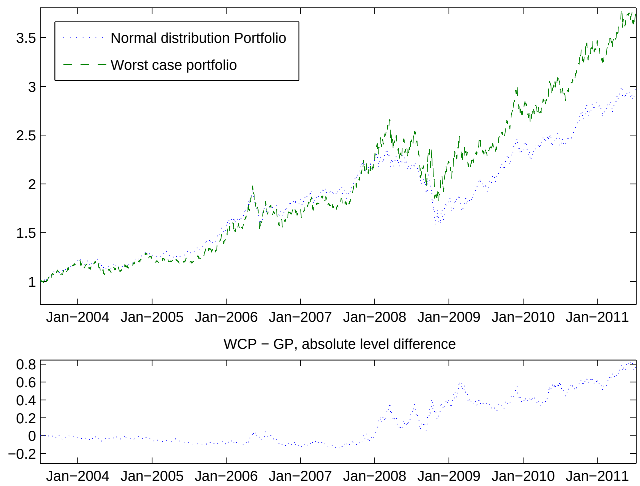


Figure 2: Satic portfolios, 'Out of Sample' Portfolio performance for $\mu = 0$ (41)

#### 4.1.2 Dynamic portfolio

We consider the case when the optimal weights of the assets are recomputed. In this more dynamic portfolio, the weights of the assets are recalibrated on a monthly basis.

At every step, we extend the in-sample calibration window by a month, then we solve problems (37) and (38) to obtain the weights and then we keep the same weights for the rest of that month. We do not adopt a moving window of calibration since we do not want to loose the information from the old crises, in our case the crises between 1998 and 2002.

So, once again we solve equations (37) and (38) with the same constraints as in Section 4.1.1 but

Table 2: Comparison of the performance of Gaussian optimal and Worst Case optimal portfolio

| $\mu$ ($10^{-3}$) | GP (I) WCP (II) | In Sample AR[1] ($10^{-3}$) | Vol[2] ($10^{-3}$) | $CVaR_{0.95}$ ($10^{-3}$) | Out of Sample AR[1] ($10^{-3}$) | Vol[2] ($10^{-3}$) | $CVaR_{0.95}$ ($10^{-3}$) | MD1[3] (%) | MD2[4] (%) | TR[5] (%) |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.00 | I | 0.00 | 6.66 | 14.4 | 0.56 | 9.00 | 22.1 | -7.44 | -31.8 | 198 |
| | II | 0.13 | 9.29 | 19.2 | 0.70 | 12.2 | 29.1 | -7.23 | -28.9 | 277 |
| 0.10 | I | 0.10 | 7.05 | 15.1 | 0.66 | 9.84 | 23.9 | -7.11 | -31.4 | 263 |
| | II | 0.13 | 9.29 | 19.2 | 0.70 | 12.2 | 29.1 | -7.23 | -28.9 | 277 |
| 0.15 | I | 0.15 | 7.56 | 16.2 | 0.71 | 10.4 | 25.3 | -7.02 | -316 | 300 |
| | II | 0.15 | 9.04 | 18.7 | 0.71 | 12.0 | 28.8 | -7.07 | -29.6 | 286 |
| 0.20 | I | 0.20 | 7.95 | 16.9 | 0.73 | 10.2 | 24.9 | -6.59 | -39.7 | 319 |
| | II | 0.20 | 8.80 | 18.1 | 0.72 | 11.1 | 26.8 | -6.89 | -35.4 | 300 |
| 0.25 | I | 0.25 | 11.3 | 24.0 | 0.74 | 11.8 | 28.5 | -7.68 | -49.8 | 313 |
| | II | 0.25 | 11.8 | 24.2 | 0.72 | 12.0 | 28.4 | -8.07 | -44.7 | 296 |

1. AR : Average return over the period
2. Vol : The volatility defined by the standard deviation
3. MD1 : Maximum draw-down, The worst return between two consecutive days
4. MD2 : Maximum draw-down, The worst return between two days within a period of maximum 6 months
5. TR : The total return from the beginning to the end of the period

with only one difference. Since we have an expanding calibration window the feasible space with respect to the constraint (41) changes as a reasult of the changing values of $\mu$ used for the Monte Carlo simulations (equation (35)). Thus, we introduce a dynamic minimum expected return. This is achieved by defining the constraint corresponding to the dynamic minimum portfolio return as

$$E(\mathbf{w}^T\mathbf{F}^{-1}(\mathbf{u})) \geq \max[\mathbf{wEq}^T\hat{\mu}, 0], \tag{42}$$

where $\hat{\mu} = (\hat{\mu}_1, ..., \hat{\mu}_n)$ is the vector of the asset return means as calculated using the calibration period and $\mathbf{wEq} = (1/n, ..., 1/n)$ i.e. all the weights are equal.

For comparison purposes, we also include a simple portfolio not based on optimization. The 'Equally weighted' portfolio (EWP) has equal positions in all assets i.e. we always use **wEq** as the weights of the portfolio.

For all three portfolios (GP, WCP and EWP) we show the out of sample performace in Figure 3 and Table 3. The observations are very similar to the example in Section 4.1.1. The CVaR and volatility of the WCP are higher than the GP but the MD2 and the overall return of the WCP is much better. Also, for the case of the EWP, the volatility and the CVaR of the portfolio lies between the other two portfolios but the portfolio performes worse with respect to the AR, TR and MD2, i.e. the EWP sustains the largest losses during the 2008 Global Recession crisis.

# 5 Summary and Conclusions

In this paper we demonstrated one way of using copulas in a portfolio optimization framework where the worst-case copula is considered. We particularly focus on the derivation of CVaR and WCVaR for copulas. In the case of WCVaR we showed how a mixture copula can be used in order to obtain a convex optimization problem.

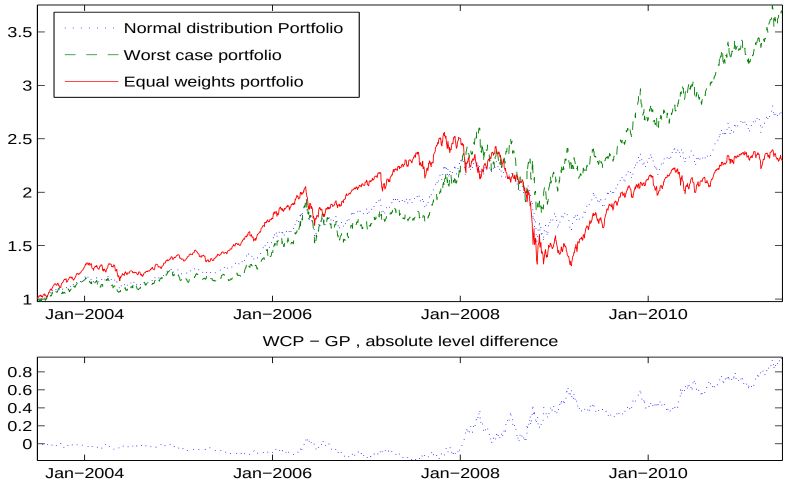


Figure 3: Dynamic portfolio performance with $\mu = \max[0, \mathbf{wEq}^T\bar{\mathbf{x}}]$

Table 3: Comparison of the performance of Gaussian optimal and Worst Case optimal portfolio

| GP (I) WCP (II) EWP (III) | Out of Sample AR[1] $(10^{-3})$ | Vol[2] $(10^{-3})$ | $\text{CVaR}_{0.95}$ $(10^{-3})$ | MD1[3] (%) | MD2[4] (%) | TR[5] (%) |
|---|---|---|---|---|---|---|
| I | 0.53 | 8.76 | 21.5 | -7.49 | -33.0 | 174 |
| II | 0.71 | 12.3 | 29.2 | -7.20 | -28.8 | 274 |
| III | 0.45 | 9.73 | 23.9 | -6.52 | -45.4 | 130 |

1. AR : Average return over the period
2. Vol : The volatility defined by the standard deviation
3. MD1 : Maximum draw-down, The worst return between two consecutive days
4. MD2 : Maximum draw-down, The worst return between two days within a period of maximum 6 months
5. TR : The total return from the beginning to the end of the period

By introducing copulas in the CVaR framework we allow more flexibility in the selection of the distribution. The most commonly used distribution for modeling multivariate dependencies is Gaussian copula. This is because of the simplicity of its construction and the availability of efficient methods for simulating from a Gaussian copula. Its disadvantages are its symmetry and the ability to only describe linear dependencies via the use of linear correlation in its structure. Symmetric behavior and linear dependencies among assets are unrealistic [1, 2, 11, 12]. We discuss alternative distribution functions in the form of copulas that can exhibit asymmetric behavior and utilize monotonic measures of dependence in their formulation. These are the three Archimedean copulas in Section 2.1 that are also easy to simulate from using the algorithms given by Melchiori [16].

The advantage of using non symmetric distribution functions was demonstrated in the numerical examples of Section 4.1. In Section 4.1 we show that for both static and dynamic strategies, for low minimum expected portfolio return the WCPs outperforms the GPs in every sector except the volatility and the CVaR. In particular during the 2008 crisis the WCPs performed more robustly than the GPs, and that was true even for high minimum expected return.

We compare the performance of dynamic portfolios with that of the EWP. The EWP performance shows that following a naive approach not considering risk is not necessarily the correct way forward. Although the EWP performed relatively adequately soon after 2002, it suffered the biggest loss during the 2008 crisis. As a result, the EWP became the worst performing portfolio among all portfolios in the numerical examples.

It seems reasonable to conclude that when optimizing a portfolio, the associated risk needs to be taken into account. All possible dependencies have to be considered in order to obtain robust results. One way of achieving this is through copulas and mixture copulas, that allow dependency systems with higher flexibility in their description than a single distribution. Further research is needed in the area of modeling to achieve better description of dependencies and risk.

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