Modelling and Managing Risk

Nicos Christofides

Alexander Christofides &

Simon Christofides June 2009

Imperial College, London SW7 2BX, UK
n.christofides@imperial.ac.uk
nicos.christofides@gmail.com
&
FINANCIAL ENGINEERING LTD
14 Osten Mews, South Kensington, London SW7 4HW, UK

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[1] n.christofides@imperial.ac.uk
[1] www.assetpricing.com

Uncertain Outcomes: One Period

$A(1), A(2), \ldots, A(\hat{m})$ are $\hat{m}$ random variables whose values today are $a(1)^0, \ldots, a(\hat{m})^0$. At the end of one period (day, week, etc.) their values have *marginal* distribution functions (DF):

$$g_m(v) = \mathbb{P}[A(m) \leqslant v], \quad m = 1, \ldots, \hat{m}.$$

The *joint* distribution function of $A(1), \ldots, A(\hat{m})$ at the end of the period, is

$$g(v_1, \ldots, v_{\hat{m}}) = \mathbb{P}[A(1) \leqslant v_1, \ldots, A(\hat{m}) \leqslant v_{\hat{m}}]$$

and encapsulates all that is known about the uncertainty in the values of $A(1), \ldots, A(\hat{m})$.

Obviously:
$$\begin{aligned} g_1(v) &= g(v, \infty, \ldots, \infty), \\ g_2(v) &= g(\infty, v, \infty, \ldots, \infty), \\ &\ldots \end{aligned}$$

[1]n.christofides@imperial.ac.uk
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The Joint DF: One Period

Because joint DFs are complicated, their application depends on our ability to simplify, or summarize their main characteristics.

Traditional descriptors of a joint DF:

- **Location**: Mean, Median, $q-$quantile (VAR at level q).
- **Dispersion**: Variance, Mean-absolute-deviation
- **Dependence**: Correlation matrix, copula functions

Algorithmic descriptors of a joint DF:

- **Mixtures of known distributions**: Normal+Normal, Normal+Uniform
- **Historical simulation**: The bootstrap
- **Model simulation**: Approximate joint DF by a model and generate scenarios from it.

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REGIME CHANGES

Even a cursory look at financial time series confirms that their behaviour is not stationary or ergodic. As a result, modelling asset-price dependence by a single joint DF is likely to lead to gross errors at some time.

In reality, markets exist in different 'regimes' depending on both endogenous and exogenous events.

The natural such partition is into [*calm*, *distressed*] regimes; or into [*calm*, *exuberant*, *distressed*] regimes.

Asset returns in calm regimes may be reasonably assumed to be Normally distributed, whereas in distressed (or exuberant) regimes extreme asset-returns are just as likely to occur as returns of average size; and the distribution is more like Uniform.

We *optimally* calibrate a model with regime-switching that naturally leads to a DF that is a mixture of distributions. In particular we examine a mixture of Normal and Uniform distributions.

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Fitting a Model to a Historical Period

The historical record of asset returns is over the time interval $[1, \dots, \hat{t}]$. At least ℓ_1 observations are needed for fitting a model to data, and at least ℓ_2 forward forecasts to determine the quality of the fit.
Hence, sub-intervals $[t', t'']$ must have $t'' - t' \geqslant \ell \equiv \ell_1 + \ell_2$, in order to fit one of the following processes:

- P_N: **A stochastic process,** with innovations drawn from a Normal distribution
- P_U: **A stochastic process,** with innovations drawn from a Uniform distribution

In both cases, we assume that the volatility follows a GARCH(1,1) process.

The 'goodness' of fit of a process to a time series interval is measured by the **Berkowitz Statistic** (BS). The lower the value of BS, the better the fit. We used $\ell_1 = 20$, $\ell_2 = 5$ working days, giving 15 measurements for the BS computation.

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Partitioning the Historical Record

Let $w_N(t',t''), w_U(t',t'')$ be the values of BS, fitting processes P_N and P_U to time series interval $[t',t'']$, respectively.
We use $w(t',t'') = \min[w_N(t',t''), w_U(t',t'')]$

Let $f(t,k)$ be the minimum error of modelling period $[1,t]$ into k calm/distressed regimes, where $t \geqslant k\ell$, otherwise $f(t,k) = \infty$. We then have:

$$f(t,k) = \min_{\ell+1 \leqslant t' \leqslant t-(k-1)\ell} [f(t-t', k-1) + w(t-t'+1, t)]$$

with initialization $f(t,1) = w(1,t), t = \ell, \ldots, \hat{t}$.
The value of $f(\hat{t},\hat{k})$ gives the minimal error in fitting calm/distressed processes to the time series, and the corresponding solution $[1, t_1, t_2, \ldots, t_{\hat{k}-1}, \hat{t}]$ gives the optimal partition of the entire period into $\hat{k}$ calm/distressed sub-periods: $[1,t_1], [t_1,t_2], \ldots, [t_{\hat{k}-1}, \hat{t}]$.

The value of $\hat{k}$ is subjective, but there should be a significant decrease in the value of $f(\hat{t}, k+1)$ over $f(\hat{t},k)$ in order to increase k further. Various heuristics can be applied to identify the “knee” of the curve.

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A PORTFOLIO: ONE PERIOD

Our object of interest is a portfolio represented by a vector Y of holdings in $\hat{n}$ *assets:* $Y = [y(1), \ldots, y(\hat{n})]$ Initially we assume Y is fixed and consider what happens to it's value $V(Y)$

If $A(1), \ldots, A(\hat{m})$ represent (perhaps unobservable) factors, then their joint DF must be converted to a joint DF of asset prices, before the effect on $V(Y)$ is considered. (Alternatively, we can represent Y in factor space and work in that space instead.)

If factors are linearly related to asset prices, the DF transformation is trivial, otherwise, simulation may be needed.

Here we assume the former, and without any further loss of generality we take the $A(m)$ to be the asset prices themselves. Thus $m = n$ and g is the joint DF of the asset prices.

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THE CONCEPT OF RISK

Related to possible loss:

(a1) What is the worst possible loss?
(a2) What is the probability of a loss?
(a3) What is the loss that would be exceeded in only $\alpha\%$ of cases? (VaR_α)
(a4) What is the expected loss in the worst $\alpha\%$ of cases? (*Tail loss*: TL_α)

Related to uncertain outcomes:

(b1) What are the two extrema of the possible outcomes?
(b2) What is the standard deviation of the possible outcomes?
(b3) What is the mean-absolute-deviation of the possible outcomes?

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COHERENT MEASURES OF RISK

Any measure of risk must satisfy some basic 'reasonable' properties in order to agree with our intuitive understanding of what constitutes risk.

If X is a random variable representing risk (say portfolio return) and $\rho(X)$ is a risk measure, then the most important such properties are:

- **Homogeneity:** If the amount invested in a portfolio is increased k-fold, the risk measure should also increase k-fold, i.e. $\rho(kX) = k\rho(X)$.
- **Subadditivity:** If two portfolios A & B are merged into C, the risk measure for C must satisfy: $\rho(X(C)) \equiv \rho(X(A) + X(B)) \leqslant \rho(X(A)) + \rho(X(B))$.
- **Monotonicity:** If the risk of a loss $\geqslant \lambda$ for portfolio A is greater or equal to that for portfolio B for all λ, then the risk measure must satisfy: $X(A) \geqslant X(B) \Rightarrow \rho(X(A)) \geqslant \rho(X(B))$
- **Translation invariance:** If a risk-free asset with return r is added to a portfolio, the risk measure is reduced by r, i.e. $\rho(X + r) = \rho(X) - r$.

Subadditivity is the property on which diversification depends. TL_α, for example, satisfies all the above requirements and is a *coherent* measure, but VaR_α only satisfies subadditivity when X is Gaussian.

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